Sophie's world: geometry and integrability

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Софья Васильевна Ковалевская, 1850–1891

Sophie's world: the story of the history of integrability

SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$
(1)\qquad
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'',\\
B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma,\\
C\frac{dr}{dt} &= (A-B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'.
\end{aligned}
$$

Les constantes $A, B, C, Mg, x_0, y_0, z_0$ qui figurent dans ces équations ont la signification suivante.

A, B, C sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

M est la masse du corps;

g l'intensité de la force de gravité;

[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.

Acta Mathematica, 1889

Addition Theorems

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\mathrm{sn}\,(x+y) = \frac{\mathrm{sn}\,x\ \mathrm{cn}\,y\ \mathrm{dn}\,y + \mathrm{sn}\,y\ \mathrm{cn}\,x\ \mathrm{dn}\,x}{1 - k^2\ \mathrm{sn}^2 x\ \mathrm{sn}^2 y}$$
$$\mathrm{cn}\,(x+y) = \frac{\mathrm{cn}\,x\ \mathrm{cn}\,y - \mathrm{sn}\,x\ \mathrm{sn}\,y\ \mathrm{dn}\,x\ \mathrm{dn}\,y}{1 - k^2\ \mathrm{sn}^2 x\ \mathrm{sn}^2 y}$$

$$\wp(x+y) = -\wp(x) - \wp(y) + \frac{1}{4}\left(\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}\right)^2$$

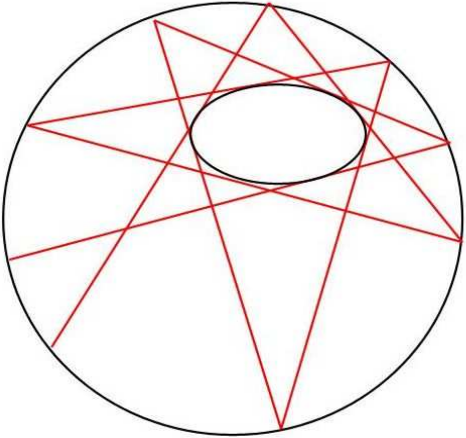
The Quantum Yang-Baxter Equation

$$R^{12}(t_1-t_2,h)R^{13}(t_1,h)R'^{23}(t_2,h) = R^{23}(t_2,h)(R^{13}(t_1,h)R^{12}(t_1-t_2,h)$$

$$R^{ij}(t,h) : V \otimes V \otimes V \to V \otimes V \otimes V$$

t – spectral parameter
h – Planck constant

The Euler-Chasles correspondence



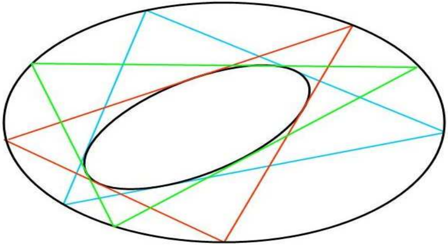
$$E : ax^2y^2 + b(x^2y + xy^2) + c(x^2 + y^2) + 2dxy + e(x + y) + f = 0$$

The Euler-Chasles correspondence

The Euler equation

$$\frac{dx}{\sqrt{p_4(x)}} \pm \frac{dy}{\sqrt{p_4(y)}} = 0$$

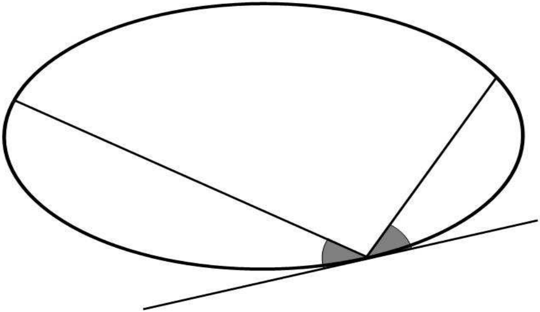
Poncelet theorem for triangles



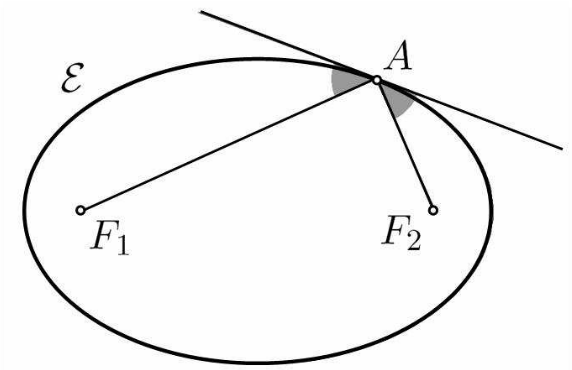
Elliptical Billiard

Billiard within ellipse

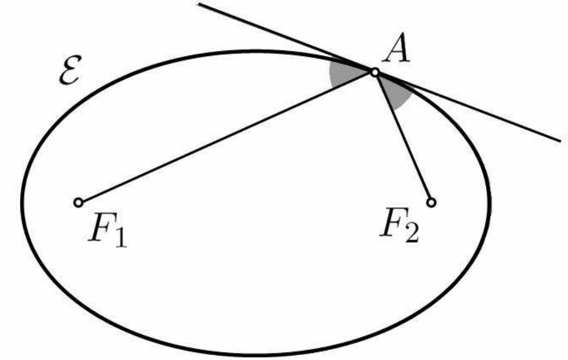
A trajectory of a billirad within an ellipse is a polygonal line with vertices on the ellipse, such that successive edges satisfy the billiard reflection law: the edges form equal angles with the to the ellipse at the common vertex.



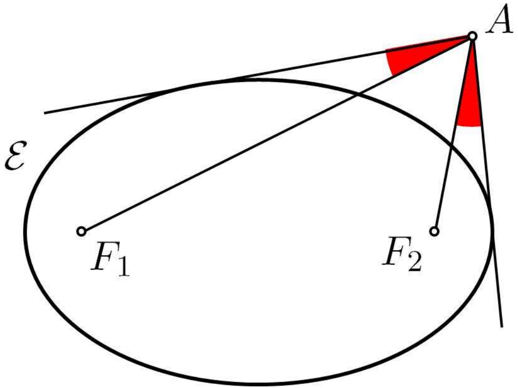
Focal property of ellipses



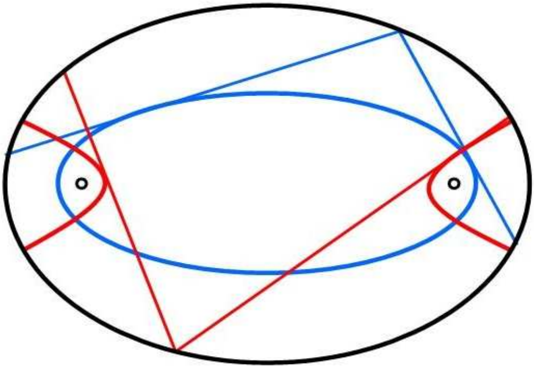
Focal property of ellipses



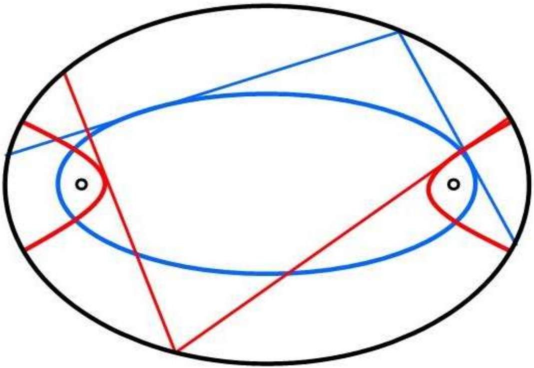
Focal property of ellipses



Caustics of billiard trajectories

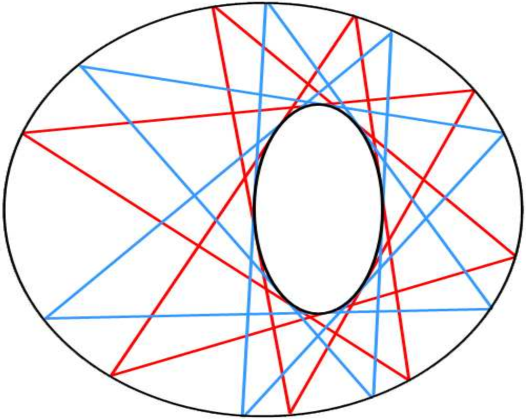


Caustics of billiard trajectories



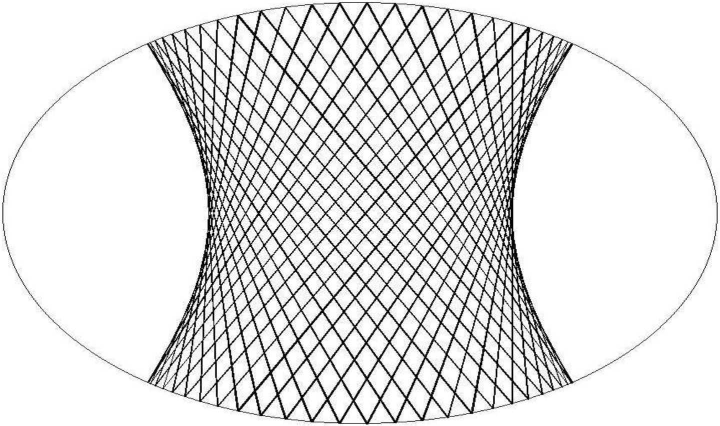
Poncelet theorem (Jean Victor Poncelet, 1813.)

Let $\mathcal{C}$ and $\mathcal{D}$ be two given conics in the plane. Suppose there exists a closed polygonal line inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{D}$. Then, there are infinitely many such polygonal lines and all of them have the same number of edges. Moreover, every point of the conic $\mathcal{C}$ is a vertex of one of these lines.

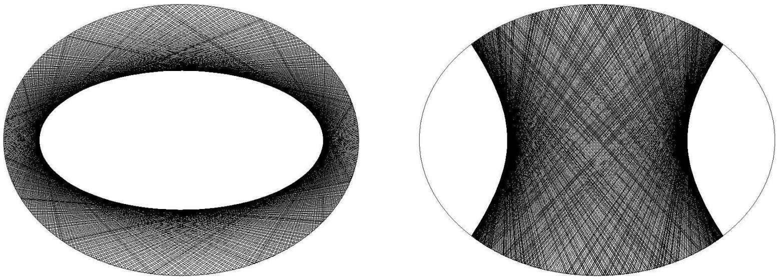


Mechanical interpretation of the Poncelet theorem

Let us consider closed trajectory of billiard system within ellipse $\mathcal{E}$. Then every billiard trajectory within $\mathcal{E}$, which has the same caustic as the given closed one, is also closed. Moreover, all these trajectories are closed with the same number of reflections at $\mathcal{E}$.



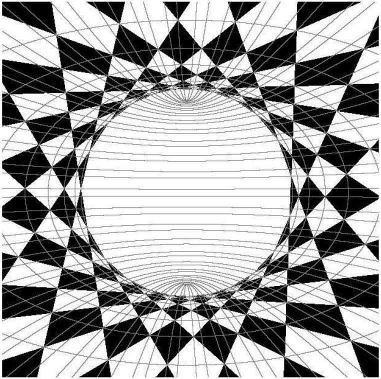
Непериодичне билијарске трајекторије



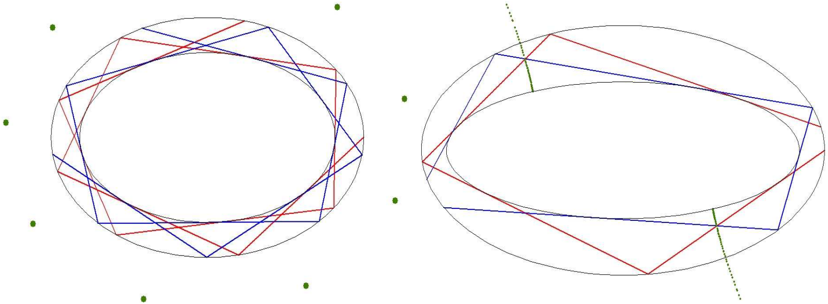
Generalization of the Darboux theorem

Theorem

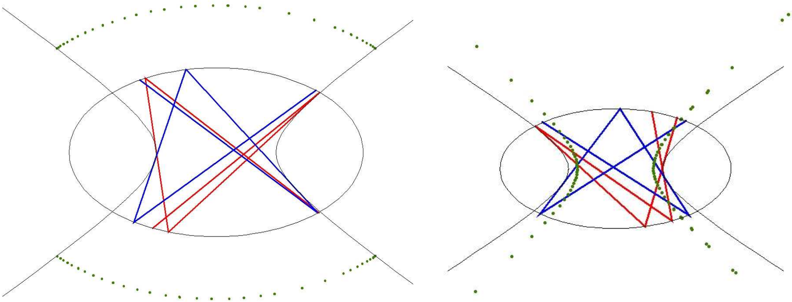
Let $\mathcal{E}$ be an ellipse in $\mathbf{E}^2$ and $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories $\mathcal{E}$, sharing the same caustic. Then all the points $a_m \cap b_m$ $(m \in \mathbf{Z})$ belong to one conic $\mathcal{K}$, confocal with $\mathcal{E}$.



Moreover, under the additional assumption that the caustic is an ellipse, we have:
if both trajectories are winding in the same direction about the caustic, then $\mathcal{K}$ is also an ellipse; if the trajectories are winding in opposite directions, then $\mathcal{K}$ is a hyperbola.



For a hyperbola as a caustic, it holds:
if segments a_m, b_m intersect the long axis of $\mathcal{E}$ in the same direction, then $\mathcal{K}$ is a hyperbola, otherwise it is an ellipse.



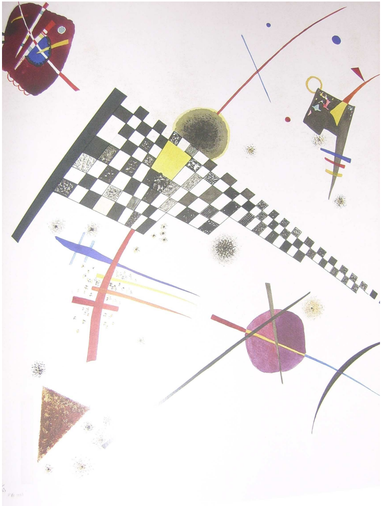
Grids in arbitrary dimension

Theorem

Let $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories within the ellipsoid $\mathcal{E}$ in $\mathbf{E}^d$, sharing the same $d-1$ caustics. Suppose the pair (a_0, b_0) is s-skew, and that by the sequence of reflections on quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$ the minimal billiard trajectory connecting a_0 to b_0 is realized.

Then, each pair (a_m, b_m) is s-skew, and the minimal billiard trajectory connecting these two lines is determined by the sequence of reflections on the same quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$.

Kandinsky, *Grid* 1923.



Pencil of conics

Two conics and tangential pencil

$$
\begin{aligned}
C_1 &: a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0 \\
C_2 &: w_2^2 - 4 w_1 w_3 = 0
\end{aligned}
$$

Coordinate pencil

$$F(s, z_1, z_2, z_3) := \det M(s, z_1, z_2, z_3) = 0$$

$$M(s, z_1, z_2, z_3) = \begin{bmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & a_0 & a_1 & a_5 - 2s \\ z_2 & a_1 & a_2 + s & a_3 \\ z_3 & a_5 - 2s & a_3 & a_4 \end{bmatrix}$$

$$F := H + Ks + Ls^2 = 0$$

Darboux coordinates

$$t_{C_2}(\ell_0) : z_1\ell_0^2 + z_2\ell_0 + z_3 = 0$$
$$\hat{z}_1\ell^2 + 2\hat{z}_2\ell + \hat{z}_3 = 0$$
$$\hat{z}_1 = 1, \quad \hat{z}_2 = -\frac{x_1 + x_2}{2}, \quad \hat{z}_3 = x_1x_2$$

$$\begin{aligned}
F(s, x_1, x_2) &= L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2)\\
H(x_1, x_2) &= (a_1^2 - a_0a_2)x_1^2x_2^2 + (a_0a_3 - a_5a_1)x_1x_2(x_1 + x_2)\\
&\quad + (a_5^2 - a_0a_4)(x_1^2 + x_2^2) + (2(a_5a_2 - a_1a_3) + \frac{1}{2}(a_5^2 - a_0a_4)x_1x_2\\
&\quad + (a_1a_4 - a_3a_5))(x_1 + x_2) + a_3^2 - a_2a_4\\
K(x_1, x_2) &= -a_0x_1^2x_2^2 + 2a_1x_1x_2(x_1 + x_2) - a_5(x_1^2 + x_2^2) - 4a_2x_1x_2\\
&\quad + 2a_3(x_1 + x_2) - a_4\\
L(x_1, x_2) &= (x_1 - x_2)^2
\end{aligned}$$

Theorem

(i) There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial F in s as a polynomial in variables x_1 and x_2 separates the variables:

$$\mathcal{D}_s(F)(x_1, x_2) = P(x_1)P(x_2). \tag{1}$$

(ii) There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial F in x_2 as a polynomial in variables x_1 and s separates the variables:

$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s)P(x_1). \tag{2}$$

Due to the symmetry between x_1 and x_2 the last statement remains valid after exchanging the places of x_1 and x_2.

Lemma

Given a polynomial $S = S(x, y, z)$ of the second degree in each of its variables in the form:

$$S(x, y, z) = A(y, z)x^2 + 2B(y, z)x + C(y, z).$$

If there are polynomials P_1 and P_2 of the fourth degree such that

$$B(y, z)^2 - A(y, z)C(y, z) = P_1(y)P_2(z), \tag{3}$$

then there exists a polynomial f such that

$$\mathcal{D}_y S(x, z) = f(x)P_2(z), \quad D_z S(x, y) = f(x)P_1(y).$$

Discriminantly separable polynoiamls – definition

For a polynomial $F(x_1, \dots, x_n)$ we say that it is discriminantly separable if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is symmetrically discriminantly separable if

$$f_2 = f_3 = \dots = f_n,$$

while it is strongly discriminatly separable if

$$f_1 = f_2 = f_3 = \dots = f_n.$$

It is weakly discriminantly separable if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

Theorem

Given a polynomial $F(s, x_1, x_2)$ of the second degree in each of the variables s, x_1, x_2 of the form

$$F = s^2 A(x_1, x_2) + 2B(x_1, x_2)s + C(x_1, x_2).$$

Denote by T_{B^2-AC} a 5×5 matrix such that

$$(B^2 - AC)(x_1, x_2) = \sum_{j=1}^{5} \sum_{i=1}^{5} T^{ij}_{B^2-AC} x_1^{i-1} x_2^{j-1}.$$

Then, polynomial F is discriminantly separable if and only if

$$\mathrm{rank}\, T_{B^2-AC} = 1.$$

Geometric interpretation of the Kowalevski fundamental equation

$$Q(w, x_1, x_2) := (x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2) = 0$$

$$\begin{aligned} R(x_1, x_2) = & - x_1^2 x_2^2 + 6\ell_1 x_1 x_2 + 2\ell c(x_1 + x_2) + c^2 - k^2 \\ R_1(x_1, x_2) = & - 6\ell_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4c\ell x_1 x_2 (x_1 + x_2) \\ & + 6\ell_1 (c^2 - k^2) - 4c^2 \ell^2 \end{aligned}$$

$$a_0 = -2 \qquad a_1 = 0 \qquad a_5 = 0$$

$$a_2 = 3\ell_1 \qquad a_3 = -2c\ell \qquad a_4 = 2(c^2 - k^2)$$

Geometric interpretation of the Kowalevski fundamental equation

Theorem

The Kowalevski fundamental equation represents a point pencil of conics given by their tangential equations

$$\hat{C}_1 : -2w_1^2 + 3l_1 w_2^2 + 2(c^2 - k^2)w_3^2 - 4clw_2 w_3 = 0;$$
$$C_2 : w_2^2 - 4w_1 w_3 = 0.$$

The Kowalevski variables w, x_1, x_2 in this geometric settings are the pencil parameter, and the Darboux coordinates with respect to the conic C_2 respectively.

Multi-valued Buchstaber-Novikov groups

n-valued group on X

$$m \ : \ X \times X \to (X)^n, \qquad m(x,y) = x * y = [z_1, \dots, z_n]$$

$(X)^n$ — symmetric n-th power of X

Associativity

Equality of two n^2-sets:

$$[x * (y * z)_1, \dots, x * (y * z)_n] \quad \text{и} \quad [(x * y)_1 * z, \dots, (x * y)_n * z]$$

for every triplet $(x, y, z) \in X^3$.

Unity e

$e * x = x * e = [x, \dots, x]$ for each $x \in X$.

Inverse $\mathrm{inv} : X \to X$

$e \in \mathrm{inv}(x) * x$, $e \in x * \mathrm{inv}(x)$ for each $x \in X$.

Multi-valued Buchstaber-Novikov groups

Action of n-valued group X on the set Y

$$\phi \ : \ X \times Y \to (Y)^n$$
$$\phi(x, y) = x \circ y$$

Two n^2-multi-subsets in Y:

$$x_1 \circ (x_2 \circ y) \quad \text{и} \quad (x_1 * x_2) \circ y$$

are equal for every triplet $x_1, x_2 \in X$, $y \in Y$.
Additionally , we assume:

$$e \circ y = [y, \dots, y]$$

for each $y \in Y$.

Two-valued group on $\mathbf{CP}^1$

The equation of a pencil

$F(s, x_1, x_2) = 0$

Isomorphic elliptic curves

$$\Gamma_1 \; : \; y^2 = P(x) \qquad \deg P = 4$$
$$\Gamma_2 \; : \; t^2 = J(s) \qquad \deg J = 3$$

Canonical equation of the curve Γ_2

$$\Gamma_2 \; : \; t^2 = J'(s) = 4s^3 - g_2 s - g_3$$

Birational morphism of curves $\psi \; : \; \Gamma_2 \to \Gamma_1$

Induced by fractional-linear mapping $\hat{\psi}$ which maps zeros of the polynomial J' and ∞ to the four zeros of the polynomial P.

Double-valued group on $\mathbf{CP}^1$

There is a group structure on the cubic Γ_2. Together with its soubgroup $\mathbf{Z}_2$, it defines the standrad double-valued group structure on $\mathbf{CP}^1$:

$$s_1 *_c s_2 = \left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right],$$

where $t_i = J'(s_i)$, $i = 1, 2$.

Theorem

The general pencil equation after fractional-linear transformations

$$F(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

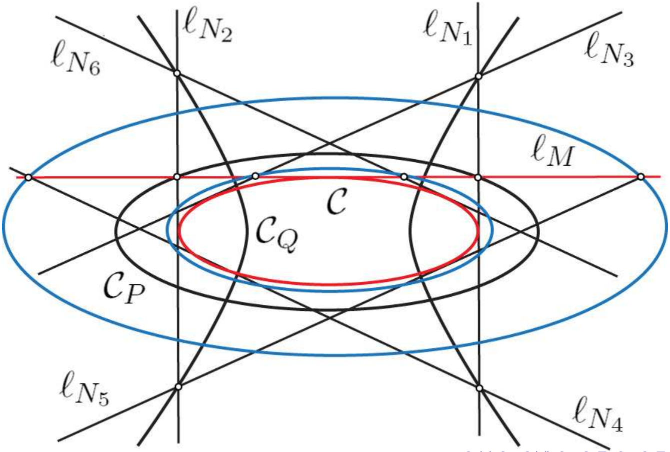
defines the double valued coset group structure $(\Gamma_2, \mathbb{Z}_2)$.

$$
\begin{tikzcd}
\mathbb{C}^4 \arrow[r, "i_{\Gamma_1}\times i_{\Gamma_1}\times m"] \arrow[d, "i_{\Gamma_1}\times i_{\Gamma_1}\times id\times id"'] \arrow[dr, "i_a\times i_a\times m"] & \Gamma_1\times\Gamma_1\times\mathbb{C} \arrow[r, "\psi^{-1}\times\psi^{-1}\times id"] \arrow[d, "p_1\times p_1\times id"] & \Gamma_2\times\Gamma_2\times\mathbb{C} \arrow[ddl, "p_1\times p_1\times id"] \\
\Gamma_1\times\Gamma_1\times\mathbb{C}\times\mathbb{C} \arrow[d, "\varphi_1\times\varphi_2"'] & \mathbb{CP}^1\times\mathbb{C} \arrow[d, "\hat{\psi}^{-1}\times\hat{\psi}^{-1}\times id"'] & \\
\mathbb{C}\times\mathbb{C} \arrow[d, "m_2"'] & \mathbb{CP}^1\times\mathbb{C} \arrow[d, "m_c\times\tau_c"'] & \\
\mathbb{CP}^2 & \mathbb{CP}^2\times\mathbb{C}/\sim \arrow[l, "f"'] &
\end{tikzcd}
$$

Double-valued group $\mathbf{CP}^1$

Theorem

Associativity conditions for the group structure of the double-valued coset group $(\Gamma_2, \mathbb{Z}_2)$ and for its action on Γ_1 are equivalent to the great Poncelet theorem for a triangle.



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