Hamiltonian approach to Dp-brane noncommutativity

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Outline of the talk

Basic facts 1

1. Strings are object with one spatial dimension. During motion string sweeps a two dimensional surface called world sheet, parametrized by timelike parameter τ and spacelike one $\sigma \in [0\,,\pi]$. There are open and closed strings.
2. Open string endpoints can be forced to move along Dp-branes by appropriate choice of boundary conditions. Dp-brane is an object with p spatial dimensions which satisfies Dirichlet boundary conditions.
3. Demanding presence of fermions in theory, we obtain superstring theory.

Basic facts 2

1. There are two standard approaches to superstring theory: NSR (Neveu-Schwarz-Ramond) (world sheet supersymmetry) and GS (Green-Schwarz) (space-time supersymmetry).
2. New approach has been recently developed - pure spinor formalism, (N. Berkovits, hep-th/0001035). It combines advantages of NSR and GS formalisms and avoid their weaknesses.
3. There are five consistent superstring theories.

Superstring theories

1. Type I
   - Unoriented open and closed strings, $N = 1$ supersymmetry, gauge symmetry group $SO(32)$.
2. Type IIA
   - Closed oriented and open strings, $N = 2$ supersymmetry, nonchiral.
3. Type IIB
   - Closed oriented and open strings, $N = 2$ supersymmetry, chiral.
4. Two heterotic theories
   - Closed oriented strings, $N = 1$ supersymmetry, symmetry group either $SO(32)$ or $E_8 \times E_8$.

Boundary conditions in canonical formalism

- As time translation generator Hamiltonian H_c must have well defined functional derivatives with respect to coordinates x^μ and their canonically conjugated momenta π_μ

$$\delta H_c = \delta H_c^{(R)} - \gamma_\mu^{(0)} \delta x^\mu \Big|_0^\pi .$$

- The first term is so called regular term. It does not contain τ and σ derivatives of coordinates and momenta variations.
- The second term has to be zero and we obtain boundary conditions.
- We obtain the same result in Lagrangian formalism from $\delta S = 0$.

Sorts of boundary conditions

- If the coordinate variations δx^μ are arbitrary at the string endpoints, we talk about Neumann boundary conditions

$$\gamma_\mu^{(0)}\Big|_0 = \gamma_\mu^{(0)}\Big|_\pi = 0\,.$$

- If the coordinates are fixed at the string endpoints

$$\delta x^\mu\Big|_0 = \delta x^\mu\Big|_\pi = 0\,,$$

then we have Dirichlet boundary conditions.

Boundary conditions as canonical constraints

- Boundary conditions are treated as canonical constraints.
- Then we perform consistency procedure. Let $\Lambda^{(0)}$ be a constraint, then consistency of constraint demands that it is preserved in time

$$\Lambda^{(n)} \equiv \frac{d\Lambda^{(n-1)}}{d\tau} = \left\{ H_c \, , \Lambda^{(n-1)} \right\} \approx 0 \, . \quad (n = 1, 2, \dots)$$

- In all cases we will consider here, this is an infinite set of constraints. Using Taylor expansion we can rewrite this set of constraints in compact σ-dependent form

$$\Lambda(\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \Omega^{(n)}(\sigma = 0) \, , \; \tilde{\Lambda}(\sigma) = \sum_{n=0}^{\infty} \frac{(\sigma - \pi)^n}{n!} \Lambda^{(n)}(\sigma = \pi) \, .$$

Bosonic string with dilaton

- Action which describes dynamics of bosonic string in the presence of gravitational $G_{\mu\nu}(x)$, antisymmetric NS-NS field $B_{\mu\nu}(x)$ and dilaton field $\Phi(x)$ is of the form

$$S = \kappa \int_{\Sigma} d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu} + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu} \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi R^{(2)} \right\} ,$$

  where $\xi^\alpha = (\tau\, , \sigma)$ parameterizes world sheet Σ with intrinsic metric $g_{\alpha\beta}$. With $R^{(2)}$ we denote scalar curvature with respect to the metric $g_{\alpha\beta}$.

Beta functions 1

- Quantum conformal invariance is determind by the beta functions

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu \,,$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} \,,$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 \,.$$

- Theory is conformal invariant on the quantum level under the following conditions $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = 0$ and $\beta^\Phi = 0$ (or $\beta^\Phi = c = const.$). This is a consequence of the relation

$$D^\nu \beta^G_{\mu\nu} + (4\pi)^2 \kappa D_\mu \beta^\Phi = 0 \,.$$

Beta functions 2

- We will consider one particular solution of these equations

$$G_{\mu\nu}(x) = G_{\mu\nu} = const\,, B_{\mu\nu}(x) = B_{\mu\nu} = const\,,$$

$$\Phi(x) = \Phi_0 + a_\mu x^\mu\,, (a_\mu = const)\,.$$

- Sigma model becomes conformal field theory with central charge c. There are two possibilities: $c = 0$, or $c \neq 0$ plus adding of Liouville term

$$S_L = -\frac{\beta^\Phi}{2(4\pi)^2\kappa}\int_\Sigma d^2\xi\sqrt{-g}R^{(2)}\frac{1}{\Delta}R^{(2)}\,, \quad \Delta = g^{\alpha\beta}\nabla_\alpha\partial_\beta\,,$$

which annihilates conformal anomaly and restores quantum conformal invariance.

Beta functions 3

- Conformal gauge condition $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$.
- There are three cases: (1) $a^2 \neq \frac{1}{\alpha}$, $\tilde{a}^2 \neq \frac{1}{\alpha}$; (2) $a^2 = \frac{1}{\alpha}$, $\tilde{a}^2 \neq \frac{1}{\alpha}$; (3) $a^2 \neq \frac{1}{\alpha}$, $\tilde{a}^2 = \frac{1}{\alpha}$. Limit $\alpha \to \infty$ gives the results for the case without Liouville term.
- Constant α is chosen in such a way that Liouville term eliminates anomaly term

$$\frac{1}{\alpha} = \frac{\beta^{\Phi}}{(4\pi\kappa)^2}\,.$$

- $\tilde{a}^2 \equiv (G^{\mu\nu}_{eff})a_\mu a_\nu$, $G^{eff}_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$.

Choice of boundary conditions

- We split x^μ ($\mu = 0, 1, 2, \ldots D-1$) into Dp-brane coordinates x^i ($i = 0, 1, \ldots p$) and orthogonal ones x^a ($a = p+1, p+2, \ldots, D-1$).
- For x^i and F we choose Neumann boundary conditions, and for x^a Dirichlet ones.
- $B_{\mu\nu} \to B_{ij}$, $a_\mu \to a_i$.

Solution of boundary conditions 1

- Initial variables are expressed in terms of their Ω even parts (effective variables), where Ω is world-sheet parity transformation $\Omega : \sigma \to -\sigma$.
- Initial coordinates depend both on effective coordinates and effective momenta.
- Presence of momenta causes noncommutativity.

Solution of boundary conditions 2

- In all three cases coordinate ${}^{\star}F = F + \frac{\alpha}{2} a_i x^i$ is commutative (in limit $\alpha \to \infty$ the role of commutative coordinate takes $a_i x^i$), while other ones are noncommutative.
- Case (1) - all constraints originating from boundary conditions are of the second class (Dirac constraints do not appear). The number of Dp-brane dimensions is unchanged.
- Case (2) - one Dirac constraint of the first class appears. The number of Dp-brane dimensions decreases.
- Case (3) - two constraints originating from boundary conditions are of the first class. The number of Dp-brane dimensions decreases.

Type IIB superstring theory in $D = 10$

- NS-NS sector: graviton $G_{\mu\nu}$, Neveu-Schwarz field $B_{\mu\nu}$ and dilaton Φ.
- NS-R sector: two gravitinos, ψ^α_μ i $\bar{\psi}^\alpha_\mu$, and two dilatinos, λ^α and $\bar{\lambda}^\alpha$. Spinors are of the same chirality.
- R-R sector: scalar C_0, two rank antisymmetric tensor $C_{\mu\nu}$ and four rank antisymmetric tensor $C_{\mu\nu\rho\sigma}$ with selfdual field strength.

Model

- We consider model without dilatinos and dilaton.
- Action for type IIB superstring theory in pure spinor formulation (ghosts free) is

$$
\begin{aligned}
S_{IIB} &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{ab}G_{\mu\nu} + \varepsilon^{ab}B_{\mu\nu}\right]\partial_a x^\mu \partial_b x^\nu \\
&+ \int_\Sigma d^2\xi \left[-\pi_\alpha(\partial_\tau - \partial_\sigma)(\theta^\alpha + \psi^\alpha_\mu x^\mu)\right] \\
&+ \int_\Sigma d^2\xi \left[(\partial_\tau + \partial_\sigma)(\bar\theta^\alpha + \bar\psi^\alpha_\mu x^\mu)\bar\pi_\alpha + \frac{1}{2\kappa}\pi_\alpha F^{\alpha\beta}\bar\pi_\beta\right]
\end{aligned}
$$

where x^μ are space-time coordinates ($\mu = 0, 1, 2, \dots 9$), and θ^α and $\bar\theta^\alpha$ are same chirality spinors.

R-R sector 1

$$F^{\alpha\beta} = \sum_{k=0}^{10} \frac{i^k}{k!} F_{(k)} \Gamma^{\alpha\beta}_{(k)} \cdot \left[\Gamma^{\alpha\beta}_{(k)} = (\Gamma^{[\mu_1 \dots \mu_k]})^{\alpha\beta} \right]$$

- Bispinor $F^{\alpha\beta}$ satisfies chirality condition $\Gamma_{11} F = -F \Gamma_{11}$, and consequently, $F_{(k)}$ (k odd) survive.
- Because of duality relation, independent tensors are $F_{(1)}$, $F_{(3)}$ and selfdual part of $F_{(5)}$.
- Massless Dirac equation for F gives, $F_{(k)} = dC_{(k-1)}$.
- Type IIB contains only potentials C_0, $C_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$.

R-R sector 2

- $$F_s^{\alpha\beta} = \frac{1}{2}(F^{\alpha\beta} + F^{\beta\alpha}) \longrightarrow C_0\,, C_{\mu\nu\rho\sigma}\;,$$
$$F_a^{\alpha\beta} = \frac{1}{2}(F^{\alpha\beta} - F^{\beta\alpha}) \longrightarrow C_{\mu\nu}\;.$$

Boundary conditions

- For coordinates x^μ we choose Neumann boundary conditions.
- In order to preserve $N = 1$ SUSY from initial $N = 2$, for fermionic coordinates we choose

$$(\theta^\alpha - \bar{\theta}^\alpha)\Big|_0^\pi = 0 \implies (\pi_\alpha - \bar{\pi}_\alpha)\Big|_0^\pi = 0\,.$$

Solution of boundary conditions

- Solving boundary conditions, we get

$$x^\mu(\sigma) = q^\mu - 2\Theta^{\mu\nu}\int_0^\sigma d\sigma_1 p_\nu + \frac{\Theta^{\mu\alpha}}{2}\int_0^\sigma d\sigma_1 (p_\alpha + \bar{p}_\alpha)\,,$$

$$\theta^\alpha(\sigma) = \eta^\alpha - \Theta^{\mu\alpha}\int_0^\sigma d\sigma_1 p_\mu - \frac{\Theta^{\alpha\beta}}{4}\int_0^\sigma d\sigma_1 (p_\beta + \bar{p}_\beta)\,,$$

$$\bar{\theta}^\alpha(\sigma) = \bar{\eta}^\alpha - \Theta^{\mu\alpha}\int_0^\sigma d\sigma_1 p_\mu - \frac{\Theta^{\alpha\beta}}{4}\int_0^\sigma d\sigma_1 (p_\beta + \bar{p}_\beta)\,,$$

where

$$\eta^\alpha \equiv \frac{1}{2}(\theta^\alpha + \Omega\bar{\theta}^\alpha)\,,\quad \bar{\eta}^\alpha \equiv \frac{1}{2}(\Omega\theta^\alpha + \bar{\theta}^\alpha)\,,$$

$$p_\alpha \equiv \pi_\alpha + \Omega\bar{\pi}_\alpha\,,\quad \bar{p}_\alpha \equiv \Omega\pi_\alpha + \bar{\pi}_\alpha\,.$$

Noncommutativity

- Using the solution, we have

$$\{x^\mu(\sigma)\,,x^\nu(\bar\sigma)\} = \Theta^{\mu\nu}\Delta(\sigma+\bar\sigma)\,,$$

$$\{x^\mu(\sigma)\,,\theta^\alpha(\bar\sigma)\} = -\frac{1}{2}\Theta^{\mu\alpha}\Delta(\sigma+\bar\sigma)\,,$$

$$\{\theta^\alpha(\sigma)\,,\bar\theta^\beta(\bar\sigma)\} = \frac{1}{4}\Theta^{\alpha\beta}\Delta(\sigma+\bar\sigma)\,.$$

Type I theory as effective one

- Putting the solution of boundary conditions into initial Lagrangian, we obtain effective one

$$\begin{aligned}\mathcal{L}^{eff} &= \frac{\kappa}{2} G^{eff}_{\mu\nu}\eta^{ab}\partial_a q^\mu \partial_b q^\nu - \pi_\alpha(\partial_\tau - \partial_\sigma)\left[\eta^\alpha + (\Psi_{eff})^\alpha_\mu q^\mu\right] \\ &+ (\partial_\tau + \partial_\sigma)\left[\bar{\eta}^\alpha + (\Psi_{eff})^\alpha_\mu q^\mu\right]\bar{\pi}_\alpha + \frac{1}{2\kappa}\pi_\alpha F^{\alpha\beta}_{eff}\bar{\pi}_\beta \,.\end{aligned}$$

- Transition from the initial $\mathcal{L}$ to effective $\mathcal{L}_{eff}$ Lagrangian is realized by changing the variables x^μ, θ^α and $\bar{\theta}^\alpha$ with q^μ, η^α and $\bar{\eta}^\alpha$,and changing the background fields

$$\begin{aligned} G_{\mu\nu} &\to G_{\mu\nu} - 4B_{\mu\rho}G^{\rho\lambda}B_{\lambda\nu}\,, \; \psi^\alpha_{+\mu} \to \psi^\alpha_{+\mu} + 2B_{\mu\rho}G^{\rho\nu}\psi^\alpha_{-\nu}\,, \\ F^{\alpha\beta}_a &\to F^{\alpha\beta}_a - \psi^\alpha_{-\mu}G^{\mu\nu}\psi^\beta_{-\nu}\,, \\ B_{\mu\nu} &\to 0\,, \qquad \psi^\alpha_{-\mu} \to 0\,, \qquad F^{\alpha\beta}_s \to 0\,. \end{aligned}$$

Conclusions

(1) Initial coordinates depend both on effective ones and effective momenta. Presence of mometa causes non(anti)commutativity of the coordinates.
(2) The number of Dp-brane dimensions depends on the relations between background fields. For particular relations between them, first class constraints appear and decrease the number of Dp-brane dimensions.
(3) Effective theory (initial one on the solution of boundary conditions) is Ω even.
(4) Effective background fields depend both on Ω even fields and Ω even combinations of the Ω odd fields.