Symbolic computation methods in cosmology and general relativity

Part I - Using Maple and GrTensorII in general relativity

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Some words before …

- **Computational physics – a new branch of physics and not only !**
- **Computational relativity – doing general relativity on the computer !**

Computer algebra for GR

Numerical simulations for GR

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Computer algebra for GR	**??????**	**Numerical simulations for GR**

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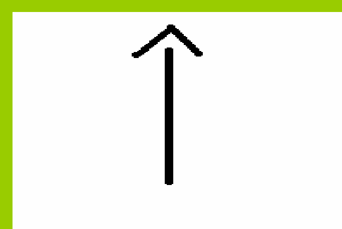
Computer algebra for GR

??????

Numerical simulations for GR



Needs a human interface !!!!

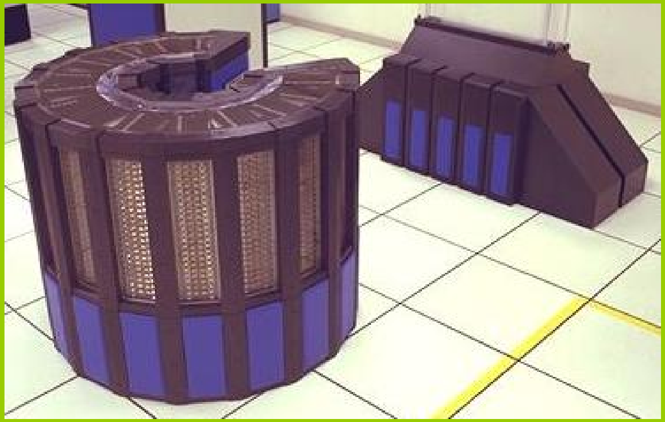
The spectacular developpment of computer technology makes possible to fill this gap in the very next future …

Some words before …

A 1956 "supercomputer" - Erma

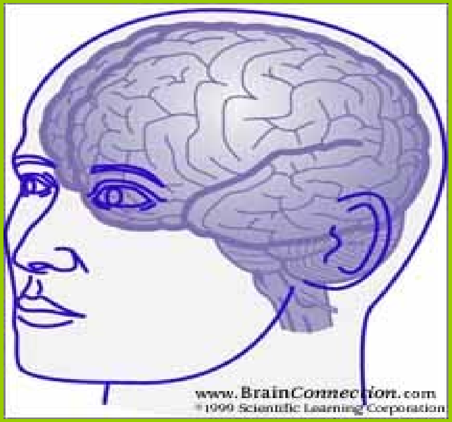
→ → → →

.... in 2009 !



A 2006 supercomputer - Cray

→ → → →



.... in 2056 ?!

Some words before …

But how about the software ? It evolved a lot from the early days of programming, so today we have :

- object oriented programming
- visual techniques
- integrated platforms (as Maple, Mathematica...)
- graphical and visualisation software
- games on the computer...

But the programming languages did not moved so fast :

- we are still using Fortran

- we are still using C, C++

Mainframe

Computer algebra - CA, is a branch of *Symbolic computation (or symbolic mathematics* - SM)

Symbolic mathematics relates to the use of computers to manipulate mathematical equations and expressions in symbolic form, as opposed to manipulating the approximations of specific numerical quantities represented by those symbols. Such a system might be used for symbolic integration or differentiation, substitution of one expression into another, simplification of an expression, etc.
(Wikipedia Free encyclopedia - http://www.wikipedia.org

Related fields and domains :

- LISP and LISP programming
- Computational logic
- Automated theorem prover or Computer-aided proof
- Artificial intellingence

But what a Computer Algebra System is ?

Mainframe

A **computer algebra system** (CAS) is a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form.

Types of expressions : The expressions manipulated by CAS typically include polynomials in multiple variables; standard functions of expressions (sine, exponential, etc.); various special functions (gamma, zeta, erf, Bessel, etc.); arbitrary functions of expressions; derivatives, integrals, sums, and products of expressions; truncated series with expressions as coefficients, matrices of expressions, and so on. (This is a recursive definition.)

The **symbolic manipulations** supported typically include

* automatic simplification, including simplification with assumptions
* substitution of symbolic or numeric values for expressions
* change of form of expressions: expanding products and powers, rewriting as partial fractions, rewriting trigonometric functions as exponentials, etc.
* differentiation with respect to one or all variables

Mainframe

The symbolic manipulations supported typically include also :

symbolic constrained and unconstrained global optimization
partial and full factorization
solution of linear and some non-linear equations over various domains
solution of some differential and difference equations
taking some limits
some indefinite and definite integration, including multidimensional integrals
integral transforms
expansion as truncated Taylor, Laurent and Puiseux series
some infinite series expansion
some series summation
matrix operations including products, inverses, etc.
display of mathematical expressions in two-dimensional mathematical form, often using typesetting systems

Mainframe

In addition, most CASs include numeric operations:

* evaluating for particular numeric values
* evaluating to high precision (bignum arithmetic), allowing for instance the evaluation of 21/3 to 10,000 digits
* numeric linear algebra
* plotting graphs and parametric plots of functions in two and three dimensions

Many also include a high level programming language, allowing users to implement their own algorithms.

The study of algorithms useful for computer algebra systems is known as computer algebra.

The run-time of numerical programs implemented in computer algebra systems is normally longer than that of equivalent programs implemented in systems such as MATLAB, GNU Octave, or directly in C, since they are programmed for full symbolic generality and thus cannot use machine numerical operations directly for most of their functions.

History

Computer algebra systems began to appear in the early 1970s, and evolved out of research into artificial intelligence, though the fields are now regarded as largely separate.

The first popular systems were Reduce, Derive, and Macsyma which are still commercially available; a copyleft version of Macsyma called Maxima is actively being maintained.

The current market leaders are Maple and Mathematica; both are commonly used by research mathematicians, scientists, and engineers.

MuPAD is a commercial system too. Some computer algebra systems focus on a specific area of application; these are typically developed in academia and free.

List of CAS systems

Proprietary - Commercial

* Derive, DoCon
* Maple, MathCad, Mathematica
* MuMATH, MuPAD
* Reduce
* WIRIS

Free / open source software

* Axiom
* dcas
* Eigenmath
* GiNaC
* Mathomatic
* Maxima
* Yacas
* SHEEP

Algebraic geometry, polynomial computations

* CoCoA
* Macaulay
* SINGULAR

List of CAS systems

	Creator	Development started	First public release	Latest stable version	Cost (USD)	Software license
Axiom	Axiom Foundation	1971	September 3, 2002	April 2005	Free	modified BSD license
Derive	Soft Warehouse	?	1988	6.1	?	Proprietary
Maple	Maplesoft	1979	1985	10	$2495+ (pro version)	Proprietary
Mathematica	Wolfram Research	1986	1988	5.2	$1880+/student<$200 (http://store.wolfram.com/catalog/)	Proprietary
Mathomatic	George Gesslein II	1986	1987	12.5.5	Free	LGPL
Maxima	Bill Schelter et al.	1967	1998	5.9.2	Free	GPL
Scilab	INRIA	?	?	3.1	Free	Open source
TI-89	Texas Instruments	?	1998	2.09	No longer in production	Proprietary
TI-89 Titanium	Texas Instruments	?	2004	3.10	MSRP $150	Proprietary
TI-92	Texas Instruments	?	1995	?	No longer in production	Proprietary
TI-92 Plus	Texas Instruments	?	1998	2.09	?No longer in production?	Proprietary
Voyage 200	Texas Instruments	?	2002	3.10	MSRP $200	Proprietary
	Creator	**Development started**	**First public release**	**Latest stable version**	**Cost (USD)**	**Software license**

Computer algebra (systems) and General Relativity

General Relativity is a theory of the dynamics of space-time, based on the differential geometry which implies long and complicated analytic computation, tensor manipulations, covariant derivatives and manny other geometrical ... ingredients !

Thus the use of CA in this field was very interesting and challenging from the very early beginning of all CAS.

CA is providing GRG new fast computational tools on the computer and on the other side, the developping of CA pushed also the research in GRG field.

Several CAS were used during the precedent decades, following the developpment of computer technology, of course. Examples : REDUCE, Maple, Maxima and even Mathematica (?!)

Some CAS were specially designed for their use in GRG : SHEEP as an example

Two CAS proved to be the most viable, having the largest spectrum of applicability and are now the most used : REDUCE and MAPLE.

Maple

Maple is a general-purpose commercial computer algebra system. It was first developed in 1981 by the Symbolic Computation Group at the University of Waterloo in Waterloo, Ontario, Canada.

Since 1988, it has been developed and sold commercially by Waterloo Maple Inc. (also known as Maplesoft), a Canadian company also based in Waterloo, Ontario. The current version is Maple 12.

Maple is an interpreted, dynamically typed programming language. As is usual with computer algebra systems, symbolic expressions are stored in memory as directed acyclic graphs.

Since Maple 6 the language has permitted variables of lexical scope.

Maple + GrTensorII

GrTensor II is a computer algebra **package** for performing calculations in the general area of differential geometry.
Authors : Peter Musgrave, Denis Pollney (!!!) and Kayll Lake

Its purpose is the calculation of tensor components on curved spacetimes specified in terms of a metric or set of basis vectors.

Though originally designed for use in the field of general relativity, GRTensorII is useful in many other fields.

GRTensor II is not a stand alone package, but requires an algebraic engine. The program was originally developed for MapleV. GRTensorII runs with all versions of Maple, Maple V Release 3 to Maple 11. A limited version (GRTensorM) has been ported to Mathematica.

GRTensor II and related software and documentation are distributed
free of charge see at http://grtensor.org

Maple + GrTensorII

The geometrical environment of GrTensorII is a Riemannian manifold with connection compatible with the riemannian metric .

Special commands and routines for calculating geometrical objects as :
- the metric and the line element (qload(), g(dn,dn),,makeg())
- Christoffel symbols and the covariant derivative (Chr(up,dn,dn))
- Ricci tensor and Ricci scalar (R(dn,dn), Ricciscalar)
- Einstein tensor (G(up,dn)) , etc.

Other facilities are available, as :
- **Several predefined metrics and possibiliy to define new ones**
- **Manipulating with indices**
- **Extracting tensor components**
- **Defining new tensors using their natural definitions**
- **Advanced simplification routines...**

Maple + GrTensorII

An example : calculating the Bianchi identities :

$$G^{i}_{j;i} = 0 \quad \text{where} \quad G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$$

Thus we have a simple sequence of Maple+ GrTensorII commands :

```
> grtw();
> qload(rob_sons);
> grdef(`bia{ ^i }:=G{ ^i ^j ;j }`);
> grcalc(bia(up));
> gralter(bia(up),simplify);
> grdisplay(bia(up));
```

If the metric is compatible with the riemannian connection the components of the Bianchi tensor must vanish. For the Robertson-Walker metric above (called "rob_sons") it gives:

For the rob_sons spacetime:
bi a(up)
bi a(up) = All components are zero

Maple + GrTensorII

For recent results in applying GrTensor II in different areas of GRG see at : http://grtensor.org inside sections Papers" and "Demonstrations" (examples : exact solutions of Einstein eqs, Newman-Penrose null tetrad formalism...)

Application to canonical ADM formalism can be found at the same website (Vulcanov demonstrations)

Application in the study of Dirac eq. on curved spacetimes with/without torsion see :

D.N. Vulcanov - Comp. Phys. Comm., vol. 154, 205 (2003)

In cosmology see :

D.N. Vulcanov, V.D. Vulcanov - Maple+GrTensor libraries for cosmology, SYNASC conference, Timisoara, 2004

Also see the contributions of Nigel Bishop !!!

End of part I

But before the break let's have a demonstration !!!