Symbolic computation methods in cosmology and general relativity

Part II - Maple+GrTensorII programs for cosmology

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Introduction

Why we need new CA packages for cosmology ?

New facts from astrophysical measurements show that the Universe is in an *accelerated expansion* - "cosmic acceleration" !

The standard model of the Universe must be reformulated !

One of the solutions is to introduce one or more scalar fields minimally coupled with the gravity to trigger this expansion.

New models are proposed daily, demanding new specific computational tools, computer algebra packages included.

This is the purpose of our Cosmo library :
to provide CA tools for theoretical cosmology, in an environment having all the cosmological parameters and functions defined !
Is entirely done using   Maple + GrTensorII !

We will describe Cosmo library in the next slides !!!

Cosmology

The Cosmo library is processing Einstein equations :

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R + \lambda g_{ij} = \frac{8\pi G}{c^4}T_{ij}$$

for the Friedmann-Robertson-Walker metric

$$ds^2 = -c^2 dt^2 + R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \; d\phi^2) \right]$$

where k=1,-1,0 and R(t) is the scale factor of the Universe

Cosmology

The matter content of the Universe is described by the stress-energy tensor :

$$T^{ij} = T^{ij}_\phi + T^{ij}_m$$

as a scalar field coupled minimally with the gravity and other matter fields separately. Thus we have :

$$T^{ij}_\phi = (p_\phi + \rho_\phi)u^i u^j + p_\phi g^{ij}$$

where, as for a perfect fluid :

$$p_\phi = -\frac{1}{2}\partial^i \partial_i \phi - \frac{1}{2}V(\phi)$$

$$\rho_\phi = -\frac{1}{2}\partial^i \partial_i \phi + \frac{1}{2}V(\phi)$$

Cosmology

For other matter content than the scalar field, the Cosmo library is providing again the stress-energy tensor as a perfect fluid one :

$$T_m^{ij} = (p + \rho)u^i u^j + pg^{ij}$$

with corresponding pressure and density variables.

The main cosmological parameters and functions, namely the Hubble "constant" and the deceleration function are :

$$H(t) = \frac{\dot{R}(t)}{R_0}$$

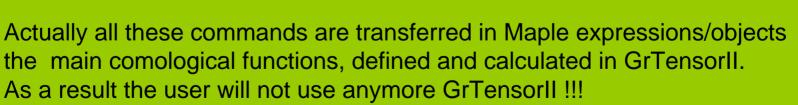$$Q(t) = -\frac{\ddot{R}(t)}{2H(t)^2 R(t)}$$

The Cosmo library

A sequence of GrTensorII and Maple commands for all this follows :

```
> restart;grtw();qload(rob_sons);
> grdef(`Scal := Phi(t)`);
> grdef(`T1{ i j } := Scal{ ,i } * Scal{ ,j } -
                        g{ i j } * (g{ ^a ^b }*Scal{ ,a }*Scal{ ,b }+ V(t) ) / 2 `);
> pphi(t):=diff(Phi(t),t)^2/2/c^2-V(t)/2;
➢epsilonphi(t):=diff(Phi(t),t)^2/2/c^2+V(t)/2;
➢grdef(`u{ i } := -c*kdelta{ i $t}`);
➢> grdef(`T2{ i j } := (epsilon(t) + p(t))*u{ i }*u{ j } + p(t) * g { i j } `);
> grdef(`T{ i j } :=T1{ i j } + T2{ i j }`);
> grdef(`cons{ i }:= T{ i ^j ;j }`); grcalc(cons(dn));
> EcuKG:=grcomponent(Box[Scal],[]) -DV(t)/2;
> grdef(`Ein{ i j } := G{ i j } - 8*Pi*G*T{ i j }/ c^4`);
> grcalc(Ein(dn,dn)); gralter(Ein(dn,dn),expand);
```

Where **pphi(t)** , **epsilonphi(t**) , **p(t)** and **epsilon(t)** are the pressure and density of the scalar field and matter field, respectively. With **cons()** we denoted the conservation law components for the total stress-energy tensor. The Klein-Gordon equation is **EcuKG** and **DV(t)** is the derivative of the potential in terms of the scalar field !!!

The Cosmo library

Then follows a sequence of boring grcomponent and gralter commands to extract the main cosmological equations. Finally we substitute the cosmological functions in the resulting MAPLE expressions:

> **Ecunr1:=expand(simplify(subs(k=K(t)*RR(t)^2,Ecunr1)));**
> **Ecunr2:=expand(simplify(subs(k=K(t)*RR(t)^2,Ecunr2)));**
> **Ecunr1:=subs(diff(RR(t),t)=H(t)*RR(t),Ecunr1);**
> **Ecunr22:=subs(diff(RR(t),t,t)=-2*H(t)^2*RR(t)*Q(t),Ecunr2);**
> **Ecunr22:=subs(diff(RR(t),t)=H(t)*RR(t),Ecunr22);**
> **Ecunr2:=subs(diff(RR(t),t)=H(t)*RR(t),Ecunr2);**
> **Ecunr2:=expand(Ecunr2);**
> **Ecunr2:=subs(diff(RR(t),t)=H(t)*RR(t),Ecunr2);**
> **Ecunr3:=subs(diff(RR(t),t)=H(t)*RR(t),Ecunr3);**
> **EcuKG:=subs(diff(RR(t),t)=H(t)*RR(t),EcuKG);**

Actually all these commands are transferred in Maple expressions/objects the main comological functions, defined and calculated in GrTensorII.
As a result the user will not use anymore GrTensorII !!!

The Cosmo library

Thus the program is providing the Klein-Gordon and the conservation law
for the scalar field, namely :

$$\frac{1}{c^2}\left[\ddot{\phi}(t) + 3H(t)\dot{\phi}(t)\right] + \frac{1}{2}DV(t) = 0$$

$$\frac{1}{c^2}\left[\ddot{\phi}(t)\dot{\phi}(t) + 3H(t)\dot{\phi}(t)^2\right] + \frac{1}{2}\dot{V}(t) + \dot{\epsilon}(t) + \\ 3H(t)(p(t) + \epsilon(t)) = 0$$

and the Friedmann equations :

$$3H(t)^2 + 3c^2K(t) - \frac{4\pi G}{c^4}\left[\dot{\phi}(t)^2 + c^2V(t) + 2c^2\epsilon(t)\right] = 0$$

$$2\dot{H}(t) + 3H(t)^2 + c^2K(t) + \frac{4\pi G}{c^4}\left[\dot{\phi}(t)^2 \\ - c^2V(t) + 2c^2p(t)\right] = 0$$

$$H(t)^2(1 - 4Q(t)) + c^2K(t) + \frac{4\pi G}{c^4}\left[\dot{\phi}(t)^2 \\ - c^2V(t) + 2c^2p(t)\right] = 0$$

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The Cosmo library

All this stuff is then saved in a library called cosmo.m and can be loaded for fast processing. As a result all the facilities of MAPLE + GrTensorII are available together with the cosmology environment described above.

The library is kept in the main directory of GrTensorII (**Grtii(6)** for Windows versions or **grii** in Unix implementations. Then it can be loaded in a Maple worksheet using read or load commands

Note : *intermediate components of the Ein(i,j) tensor are kept also (apart from Ecunr1...Ecun3 and EcuKG objects) for later use and processing. This makes easy the development of the library for more applications*

A simple example of how to use and developp the Cosmo library follows !

But first let's have a short demonstration …

A simple example

We shall ilustrate the use and development of the Cosmo library in a specificmodel using the "reverse - engineering" method for designing potentials of the scalar field.

In the **standard cosmology** the Friedmann eqs. are solved for a specific potential of the scalar field, initially prescribed from certain physical arguments, and then the time function $R(t)$ is obtained and compared with the astrophysical measurements.

In the **"reverse-engineering"** method, the function $R(t)$ is initially prescribed, as much as possible close to the measurements, and then the potential $V(t)$ is obtained from Friedmann eqs, if it is possible !

A simple example

First step : simply solve two of the above Friedmanneqs. to get :

$$V(t) = \frac{1}{4\pi}\left[\dot{H}(t) + 3H(t)^2 + \frac{2k}{R(t)^2}\right]$$

$$\dot{\phi}^2 = \frac{1}{4\pi}\left[-\dot{H}(t) + \frac{k}{R(t)^2}\right]$$

Here we have the scalar field as the only matter content of the Universe and geometrical units : c=G=1
For a DeSitter exponential expansion, namely

$$R(t) = R_0 e^{\omega t} \quad ; \quad H(t) = \omega$$

we have, after a sequence of **subs** and **simplify** commands :

$$V(t) = \frac{3\omega^2}{4\pi} + \frac{k}{2\pi e^{2\omega t}}$$

$$\dot{\phi}(t)^2 = \frac{k}{4\pi e^{2\omega t}}$$

Note : we used an intermediate Maple function for D2Phi(t) !

A simple example

Then, from the last expression we have :

$$\phi(t) = -\frac{1}{2}\frac{\sqrt{k}e^{-\omega t}}{\sqrt{\pi}\omega} + \phi_0$$

The result : equations **Ecunr1...Ecunr3** are automatically satisfied and the Klein-Gordone equation becomes :

$$\textbf{EcuKG} = \frac{\sqrt{k}\omega}{\sqrt{\pi}e^{\omega t}} + \frac{1}{2}DV(t) = 0$$

This one is used to check the calculations, solving it for the object DV(t). Therefore DV(t) must fit with the one obtained directly from the scalar field expression above, after eliminating the time...

A simple example

Final step : expressing the potential and DV(t) in terms of the scalar field, again
after a boring sequence of subs and simplify commands :

$$V(\phi(t)) = \frac{3\omega}{4\pi} + 2\omega^2(\phi(t) - \phi_0)^2$$

$$DV(\phi(t)) = 4\omega^2(\phi(t) - \phi_0)$$

We need the last expression for DV for our numerical investigations we done,
As initial data for numerical eolving the Einsten equations for the scalar field.
But this is for the moment out of the goals of this lecture.

This was a simple example, just to ilustrate how the things are working with
Cosmo library and the reverse engineering method.

Other exmples, more complicated, were processed, even with two scalar fields
included to simulate different components of the dark-matter and energy !

Here are some of examples we processed …

	$R(t)/R_0$	$\omega\alpha/B$	$V(\phi)$	$4\pi B^2$
1.	$e^{\omega t}$	$\pm e^{-\omega t}$	$\frac{3\omega^2}{4\pi} + 2\omega^2\alpha^2$	$\frac{k}{R_0^2}$
2.	$sinh(\omega t)$	$\pm ln\left(tanh(\frac{\omega t}{2})\right)$	$\frac{3\omega^2}{4\pi} + 2\left[B sinh\left(\frac{\omega}{B}\alpha\right)\right]^2$	$\frac{k}{R_0^2} + \omega^2$
3.	$cosh(\omega t)$	$\pm 2tan^{-1}(e^{\omega t})$	$\frac{3\omega^2}{4\pi} + 2\left[B sin\left(\frac{2\omega}{B}\alpha\right)\right]^2$	$\frac{k}{R_0^2} - \omega^2$
4.	t^n	$\pm\omega\, lnt$	$B^2(3n-1)e^{\pm\frac{2}{B}\alpha}$	$n; k = 0$
5.	t	$\pm\omega\, lnt$	$2B^2 e^{-\frac{2}{B}\alpha}$	$1 + \frac{k}{R_0^2}$
6.	$sin(\omega t)$	$\pm ln(tan(\frac{\omega t}{2}))$	$2\left[B cosh(\frac{\alpha\omega}{B})\right]^2 - \frac{3\omega^2}{4\pi}$	$1 + \frac{k}{R_0^2}$

Ellis –Madsen potentials. Here we denoted with R_0 the scale factor at the actual time t_0 and with α the quantity $\phi(t) - \phi_0$

Some references

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End of part II

But before the break let's have aagain a demonstration !!!