Symbolic computation methods in cosmology and general relativity

Part III - Reverse engineering method in cosmology with scalar field

Dumitru N. Vulcanov

West University of Timişoara

Theoretical and Applied Physics Dept.-“Mircea Zăgănescu”

Cosmology - remember

The Cosmo library is processing Einstein equations :

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R + \lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij}$$

for the Friedmann-Robertson-Walker metric

$$ds^2 = -c^2 dt^2 + R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \; d\phi^2) \right]$$

where k=1,-1,0 and R(t) is the scale factor of the Universe

Cosmology remember

The matter content of the Universe is described by the stress-energy tensor :

$$T^{ij} = T^{ij}_{\phi} + T^{ij}_{m}$$

as a scalar field coupled minimally with the gravity and other matter fields separately. Thus we have :

$$T^{ij}_{\phi} = (p_{\phi} + \rho_{\phi})u^i u^j + p_{\phi} g^{ij}$$

where, as for a perfect fluid :

$$p_{\phi} = -\frac{1}{2}\partial^i \partial_i \phi - \frac{1}{2}V(\phi)$$

$$\rho_{\phi} = -\frac{1}{2}\partial^i \partial_i \phi + \frac{1}{2}V(\phi)$$

Cosmology - remember

For other matter content than the scalar field, the Cosmo library is providing again the stress-energy tensor as a perfect fluid one :

$$T_m^{ij} = (p + \rho) u^i u^j + p g^{ij}$$

with corresponding pressure and density variables.

The main cosmological parameters and functions, namely the Hubble "constant" and the deceleration function are :

$$H(t) = \frac{\dot{R}(t)}{R_0}$$

$$Q(t) = -\frac{\ddot{R}(t)}{2H(t)^2 R(t)}$$

The Cosmo library remember

Cosmo library is providing the Klein-Gordon and the conservation law for the scalar field, namely :

$$\frac{1}{c^2}\left[\ddot{\phi}(t) + 3H(t)\dot{\phi}(t)\right] + \frac{1}{2}DV(t) = 0$$

$$\frac{1}{c^2}\left[\ddot{\phi}(t)\dot{\phi}(t) + 3H(t)\dot{\phi}(t)^2\right] + \frac{1}{2}\dot{V}(t) + \dot{\epsilon}(t) + 3H(t)(p(t) + \epsilon(t)) = 0$$

and the Friedmann equations :

$$3H(t)^2 + 3c^2K(t) - \frac{4\pi G}{c^4}\left[\dot{\phi}(t)^2 + c^2V(t) + 2c^2\epsilon(t)\right] = 0$$

$$2\dot{H}(t) + 3H(t)^2 + c^2K(t) + \frac{4\pi G}{c^4}\left[\dot{\phi}(t)^2 - c^2V(t) + 2c^2p(t)\right] = 0$$

$$H(t)^2(1 - 4Q(t)) + c^2K(t) + \frac{4\pi G}{c^4}\left[\dot{\phi}(t)^2 - c^2V(t) + 2c^2p(t)\right] = 0$$

Reverse engineerng remember

In the **standard cosmology** the Friedmann eqs. are solved for a specific potential of the scalar field, initially prescribed from certain physical arguments, and then the time function R(t) is obtained and compared with the astrophysical measurements.

In the **"reverse-engineering"** method, the function R(t) is initially prescribed, as much as possible close to the measurements, and then the potential V(t) is obtained from Friedmann eqs, if it is possible !

Reverse engineerng remember

	$R(t)/R_0$	$\omega\alpha/B$	$V(\phi)$	$4\pi B^2$
1.	$e^{\omega t}$	$\pm e^{-\omega t}$	$\frac{3\omega^2}{4\pi} + 2\omega^2\alpha^2$	$\frac{k}{R_0^2}$
2.	$sinh(\omega t)$	$\pm ln\left(tanh(\frac{\omega t}{2})\right)$	$\frac{3\omega^2}{4\pi} + 2\left[Bsinh\left(\frac{\omega}{B}\alpha\right)\right]^2$	$\frac{k}{R_0^2} + \omega^2$
3.	$cosh(\omega t)$	$\pm 2tan^{-1}(e^{\omega t})$	$\frac{3\omega^2}{4\pi} + 2\left[B\,sin\left(\frac{2\omega}{B}\alpha\right)\right]^2$	$\frac{k}{R_0^2} - \omega^2$
4.	t^n	$\pm\omega\ lnt$	$B^2(3n-1)e^{\pm\frac{2}{B}\alpha}$	$n; k = 0$
5.	t	$\pm\omega\ lnt$	$2B^2e^{-\frac{2}{B}\alpha}$	$1 + \frac{k}{R_0^2}$
6.	$sin(\omega t)$	$\pm ln(tan(\frac{\omega t}{2}))$	$2\left[Bcosh(\frac{\alpha\omega}{B})\right]^2 - \frac{3\omega^2}{4\pi}$	$1 + \frac{k}{R_0^2}$

Ellis-Madsen potentials….

More examples

A new type of density factor is introduced :

$$\rho = \Omega \rho_{\phi}$$

suggested in the recent literature and from the experimental measurements on the quantitative proportions between barionic, radiative, dark matter and dark energy.

Remark : it is very improbable that this density factor is constant for long time evolution !

Thus we have here an approximation - our results are good for prescribing initial data for numerical simulations !

This was the purpose of our next investigations ilustrated here in what follows

More examples

Thus, for matter as a perfect fluid we have, as usual :

$$p = (\gamma - 1)\rho = \Omega(\gamma - 1)\rho_\phi$$

where, of course, we have :

$\gamma = 1; p = 0$ dust pressureless matter

$\gamma = 4/3; p = \rho/3$ radiative matter

Following the same steps as in the previous simple example - I.e. solving the Friedman eqs, Klein-Gordon and the conservation law eqs. (Ecunr1...Ecunr3, EcuKG) we have, finally :

More examples

$$V(t) = \frac{1}{4\pi}\dot{H}(t) + \frac{3}{4\pi}H(t)^2\frac{1+\frac{1}{2}\Omega\gamma}{1+\Omega} + \frac{k}{4\pi R(t)^2}\frac{2-\Omega(1-\frac{3}{2}\gamma)}{1+\Omega}$$

$$\dot{\phi}^2 = -\frac{1}{4\pi}\dot{H}(t) - \frac{1}{4\pi}H(t)^2\frac{\frac{3}{2}\Omega\gamma}{1+\Omega} + \frac{k}{4\pi R(t)^2}\frac{1+\Omega(1-\frac{3}{2}\gamma)}{1+\Omega}$$

These were obtained after a series of solve, subs and simplify commands.

Only Maple commands manipulation !

Conclusion : Cosmo library can be used even by those nonfamiliar with GrTensorII !

More examples

$$V(t) = \frac{1}{4\pi}\dot{H}(t) + \frac{3}{4\pi}H(t)^2 \frac{1 + \frac{1}{2}\Omega\gamma}{1+\Omega} + \frac{k}{4\pi R(t)^2}\frac{2 - \Omega(1 - \frac{3}{2}\gamma)}{1+\Omega}$$

$$\dot{\phi}^2 = -\frac{1}{4\pi}\dot{H}(t) - \frac{1}{4\pi}H(t)^2 \frac{\frac{3}{2}\Omega\gamma}{1+\Omega} + \frac{k}{4\pi R(t)^2}\frac{1 + \Omega(1 - \frac{3}{2}\gamma)}{1+\Omega}$$

From now one the things are depending on how complex are the above equations. Mainly we have troubles with the second one !

Some of the examples we processed are with exact analytical solutions, some need certain approximation assumptions.

In searching initial data for numerical simulations, the last approximative solutions can be good, at least a short period after the initial time !

Linear expansion

An example with simple analytic solution : for linear expansion

$$R(t) = R_0 t$$

$$H(t) = \frac{1}{t}$$

we obtained

$$\phi(t) = \phi_0 + B' \ln t$$

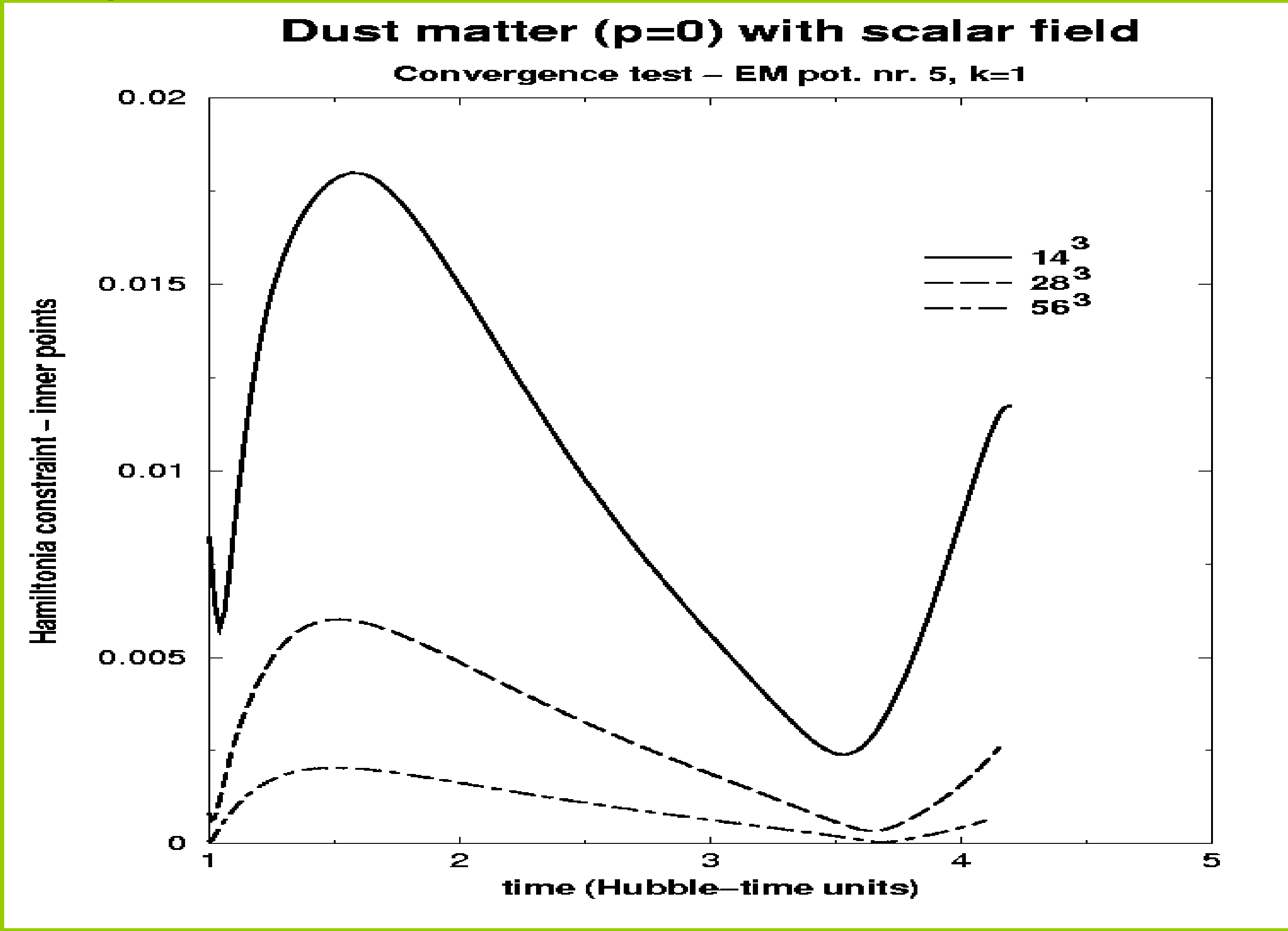
$$V(\phi) = \left[\frac{3(k+1)}{4\pi(1+\Omega)} - B'^2\right] e^{-\frac{2}{B'}(\phi(t)-\phi_0)}$$

Where

$$B' = \sqrt{\frac{1}{4\pi}(k+1)\left(1 - \frac{3}{2}\frac{\Omega}{1+\Omega}\gamma\right)}$$

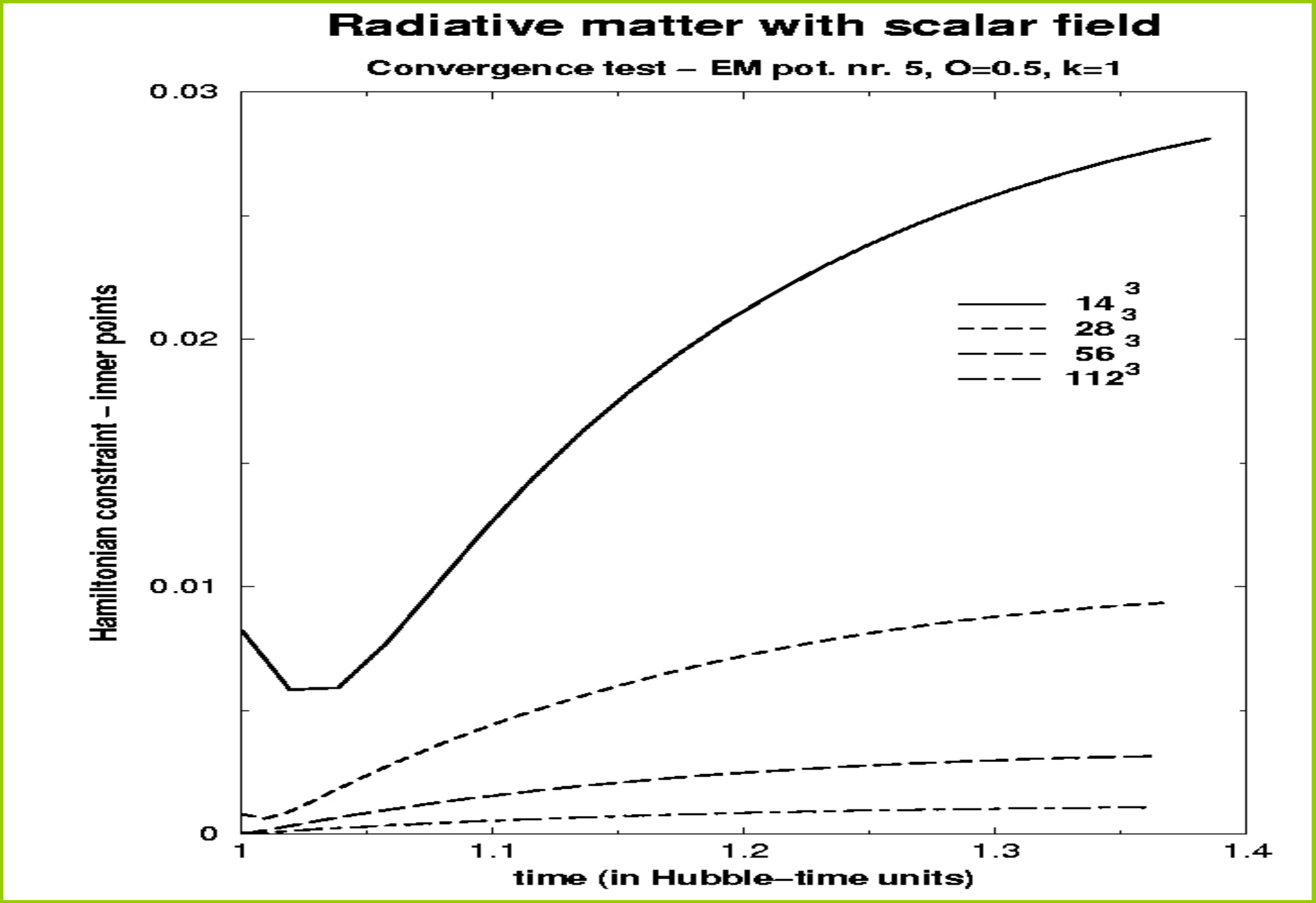
Now comes some numerical simulations with these results as initial data.
We used Cosmo thorn within the Cactus code !

Linear expansion



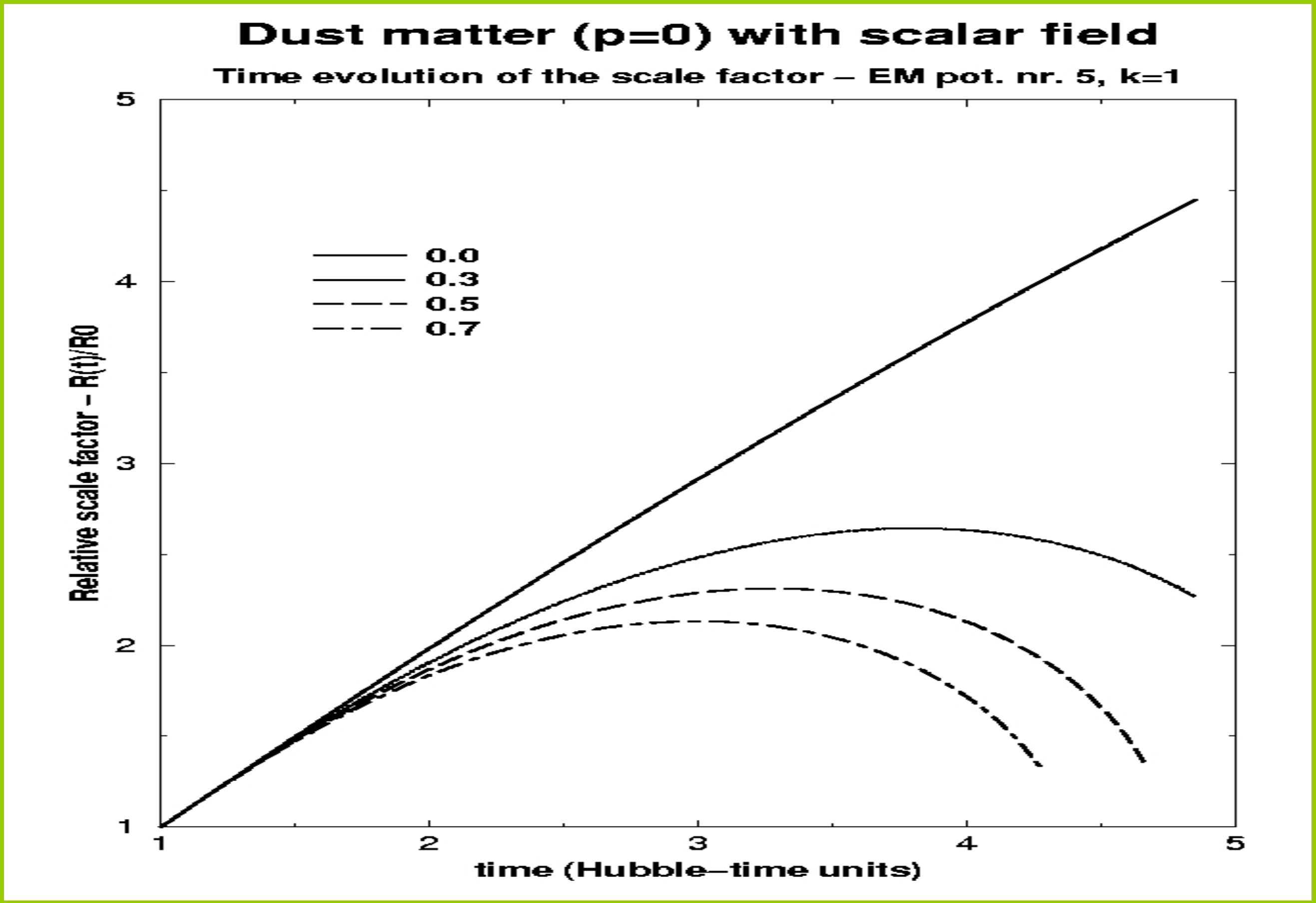
Convergence test : linear expansion with dust matter

Linear expansion



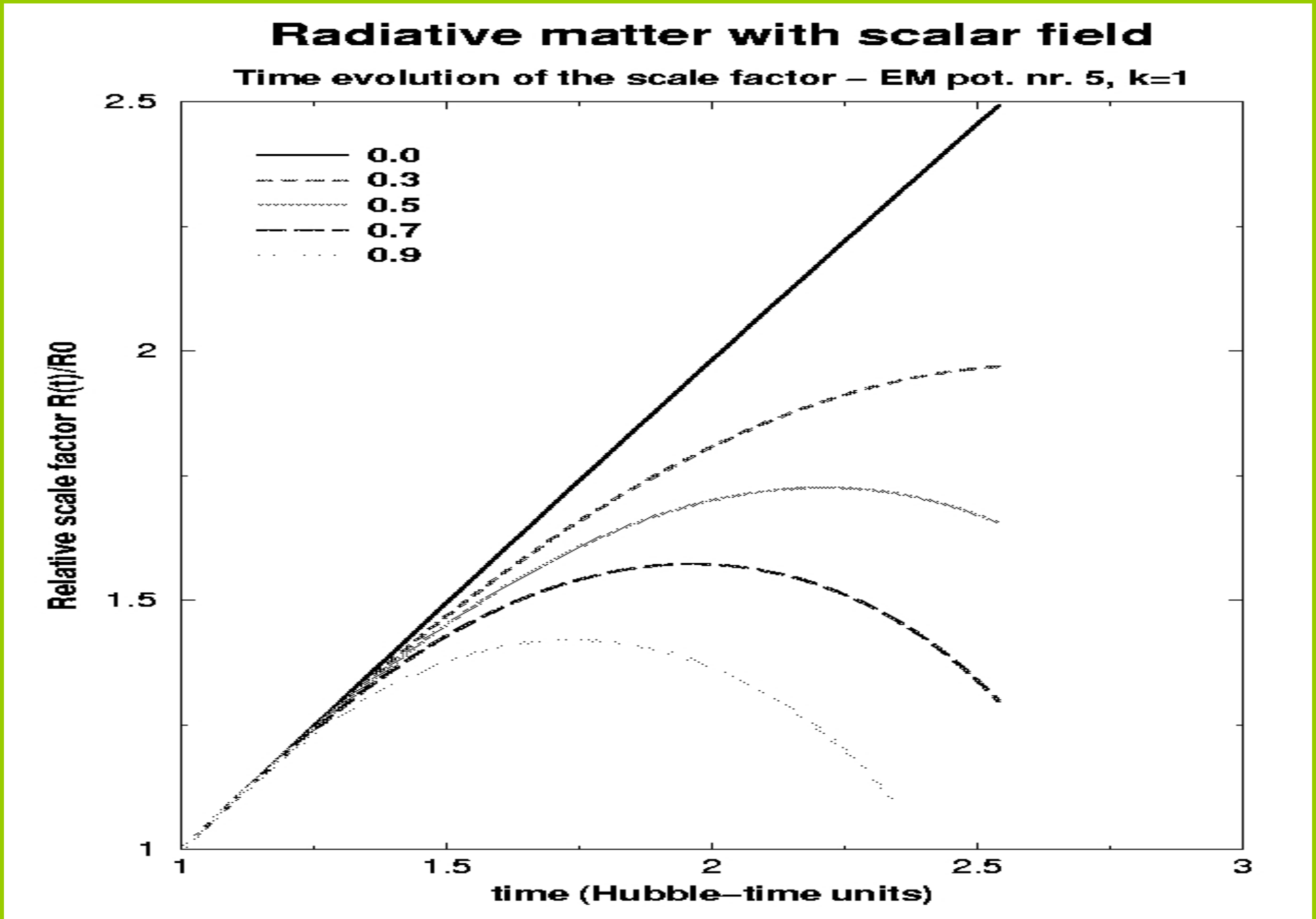
Convergence test : linear expansion with radiative matter

Linear expansion



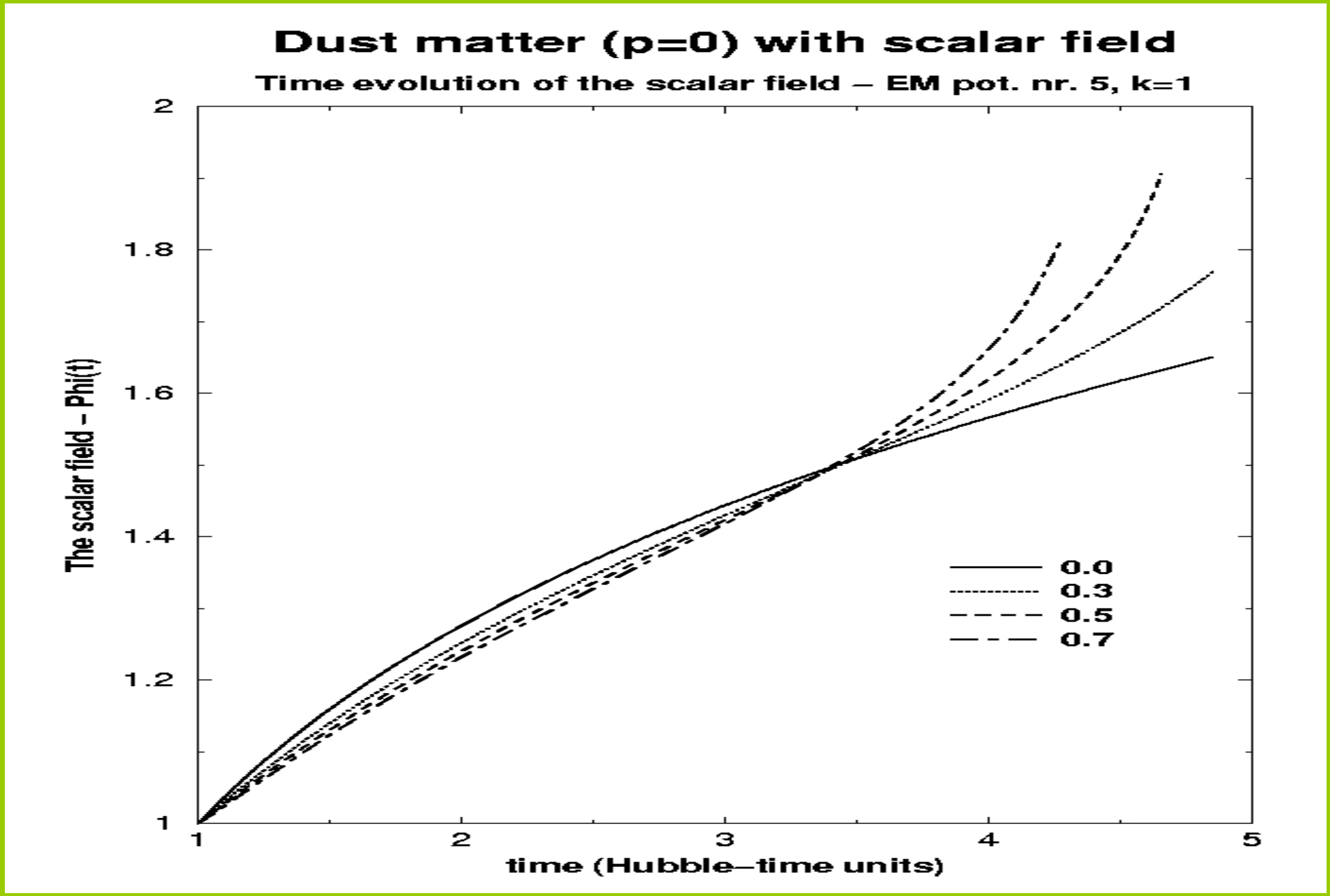
Scale factor time behavior with dust matter

Linear expansion



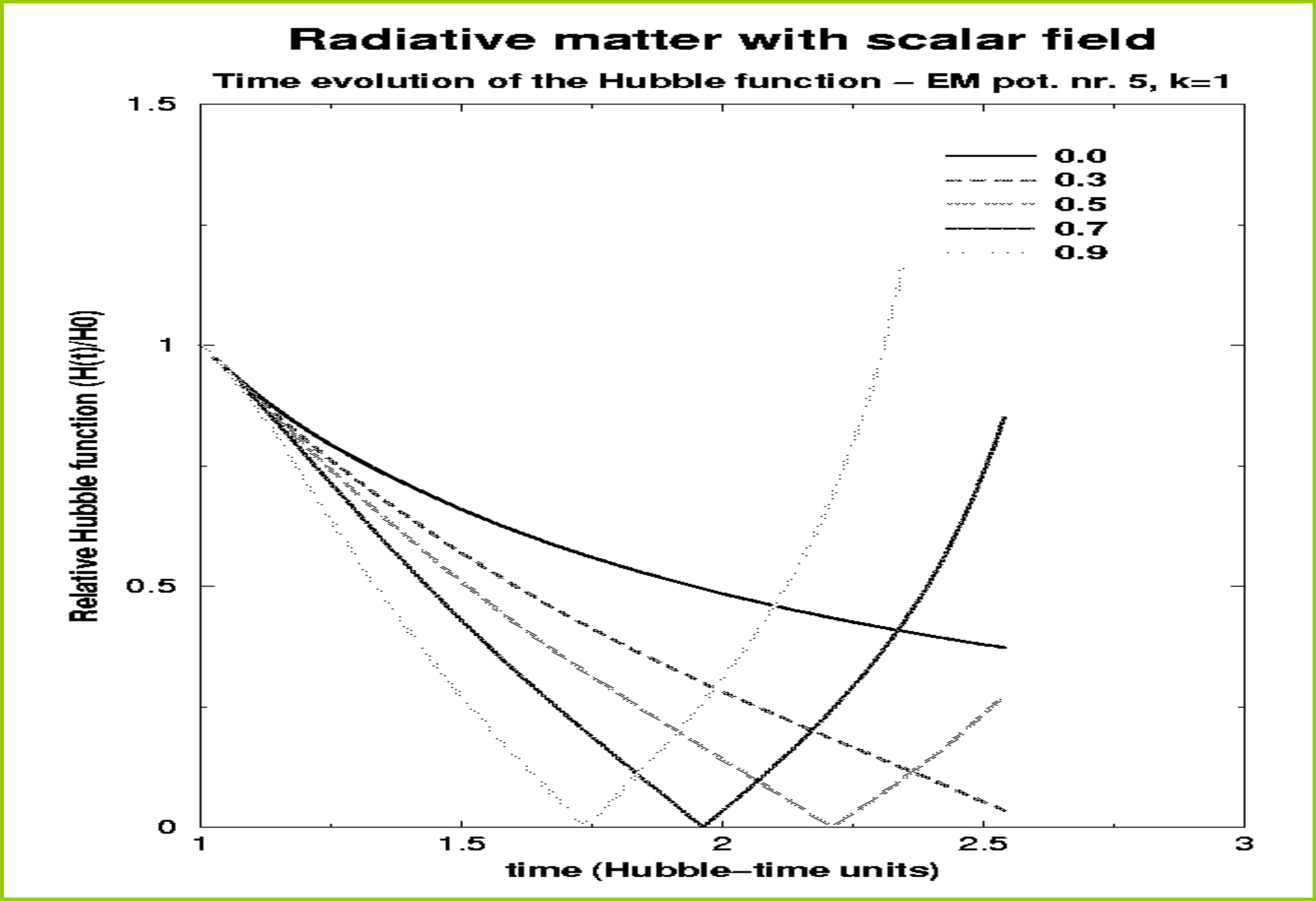
Scale factor time behavior with radiative matter

Linear expansion



Scalar field time behavior with dust matter

Linear expansion



Hubble function time behavior with radiative matter

DeSItter exponential expansion

Another example : again the DeSitter exponential expansion :

$$R(t) = R_0 e^{\omega t} \quad ; \quad H(t) = \omega$$

Here we obtained, for the closed case (k=1) :

$$\begin{aligned} \phi(t) &= -\frac{1}{2\sqrt{2\pi}}\sqrt{\frac{3\Omega}{\gamma(1+\Omega)}}\arctan\frac{\sqrt{3\Omega\gamma}}{\alpha}e^{\omega t} \\ &- \frac{1}{2\sqrt{2\pi}}\frac{\alpha}{\omega\sqrt{1+\Omega}}e^{-\omega t} + \phi_0 \end{aligned}$$

Where

$$\alpha = \sqrt{2(1+\Omega) - 3\Omega\gamma(\omega^2 e^{2\omega t} + 1)}$$

And

$$\begin{aligned} V(t) &= \frac{1}{8\pi(1+\Omega)}\left[3\omega^2(2+\Omega\gamma)\right. \\ &+ \left.(3\Omega\gamma + 2(2-\Omega))\, e^{-2\omega t}\right] \end{aligned}$$

DeSItter exponential expansion

The final step : expressing V(t) in terms of the scalar field to obtain :

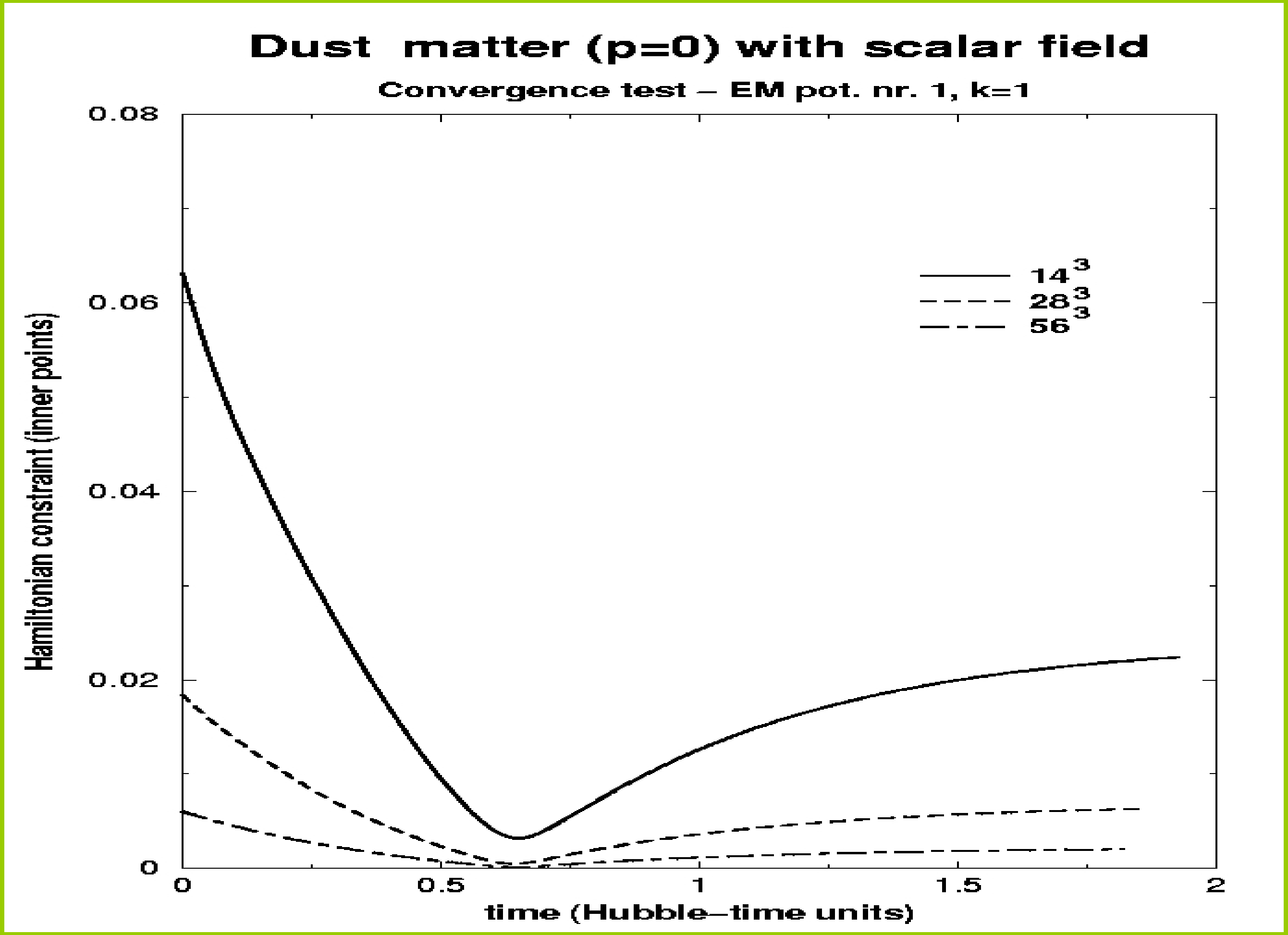
$$V(\phi) = \omega^2 \left[\frac{C}{D} (\phi(t) - \phi_0)^2 + \right.$$

$$\left. \frac{1}{2} \sqrt{\frac{3\pi\Omega\gamma}{2(1+\Omega)}} \frac{C}{D} (\phi(t) - \phi_0) + \frac{3}{2\pi D} \left(1 + C \frac{\pi^2 \Omega \gamma}{16(1+\Omega)} \right) \right]$$

where C and D have certain complicated expressions in term of the cosmological parameters.

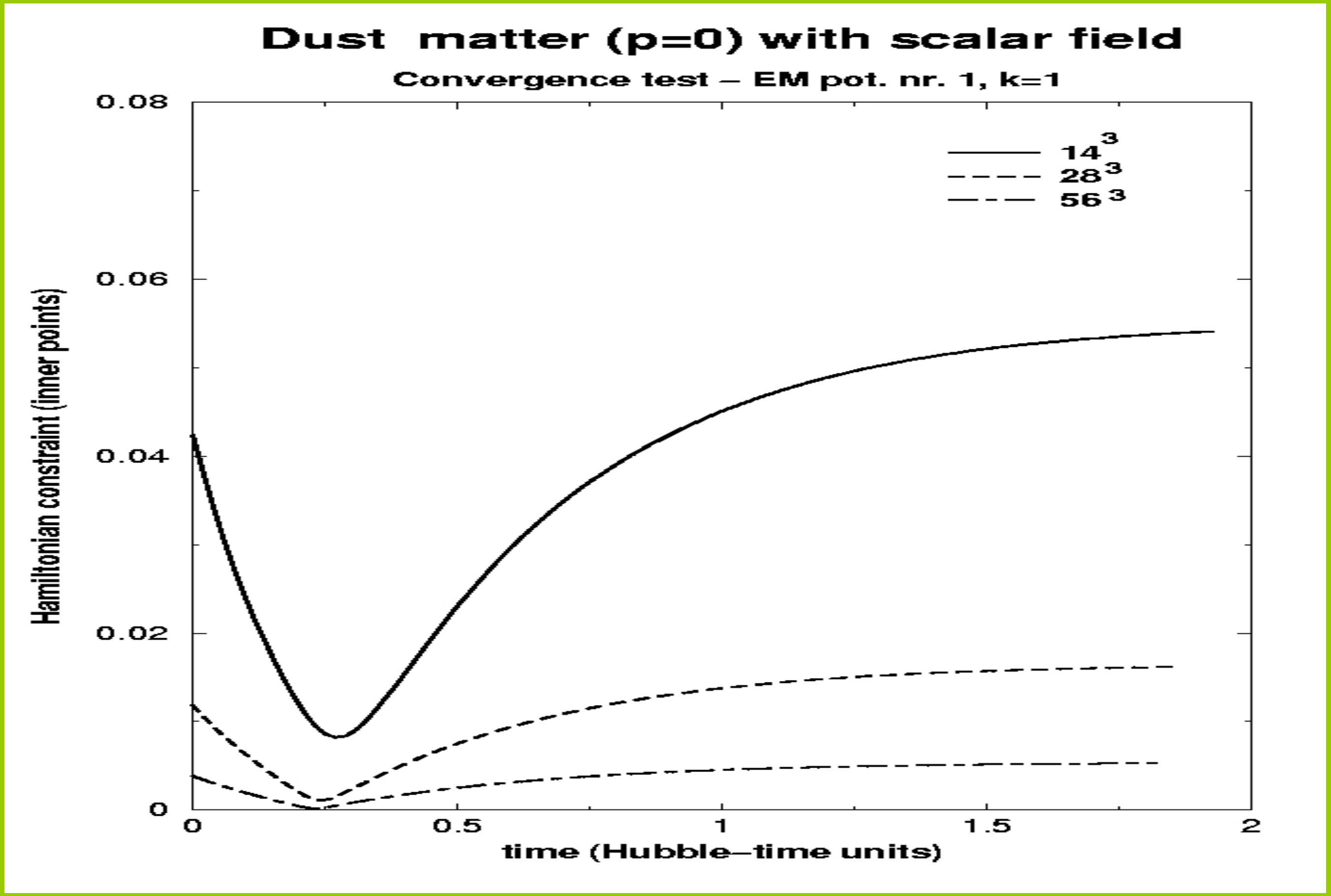
We investigated again the convergence of the numerical simulations having these results as initial data :

DeSItter exponential expansion



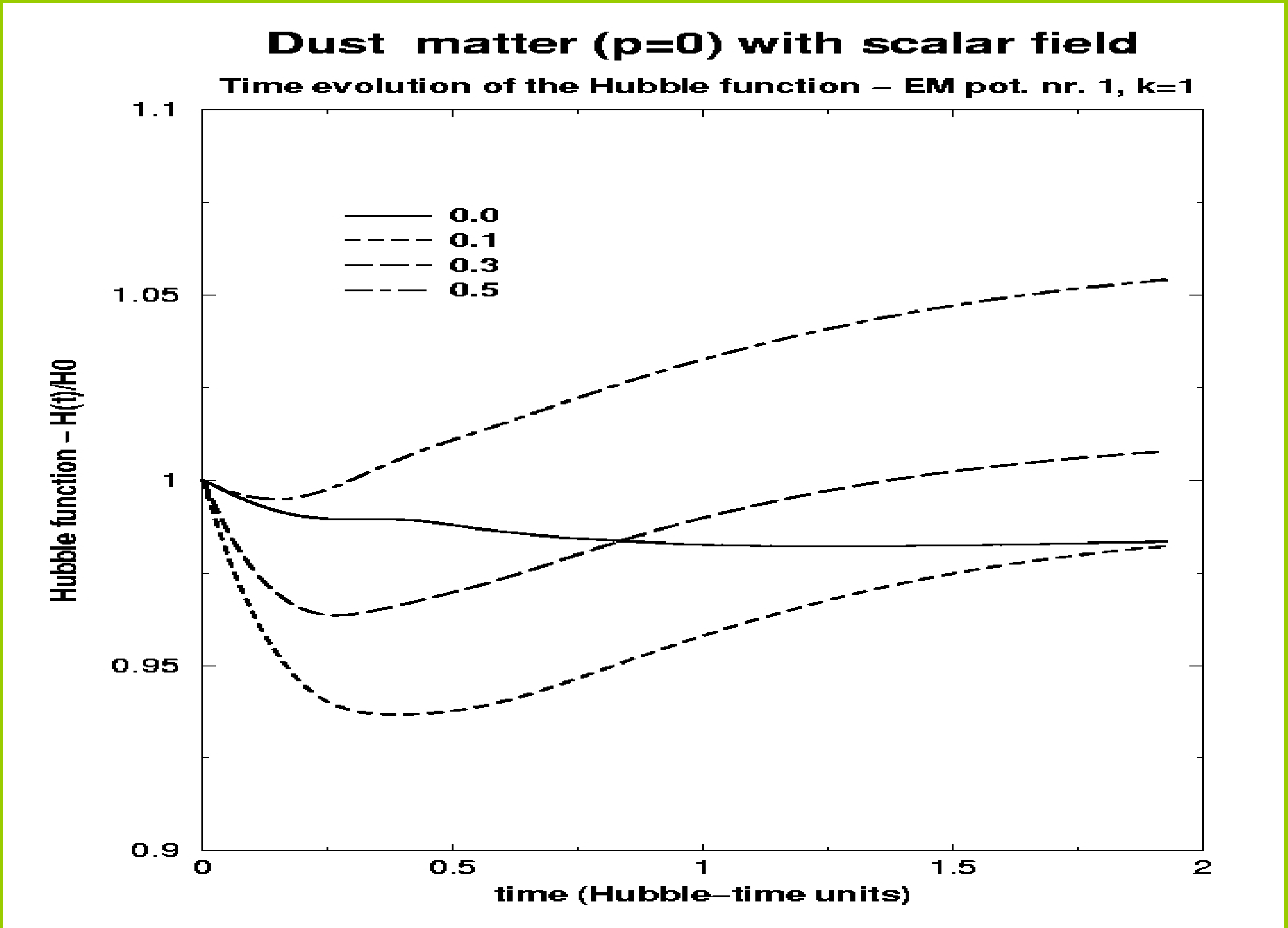
Convergence test for p=0 and Omega = 0.1

DeSItter exponential expansion



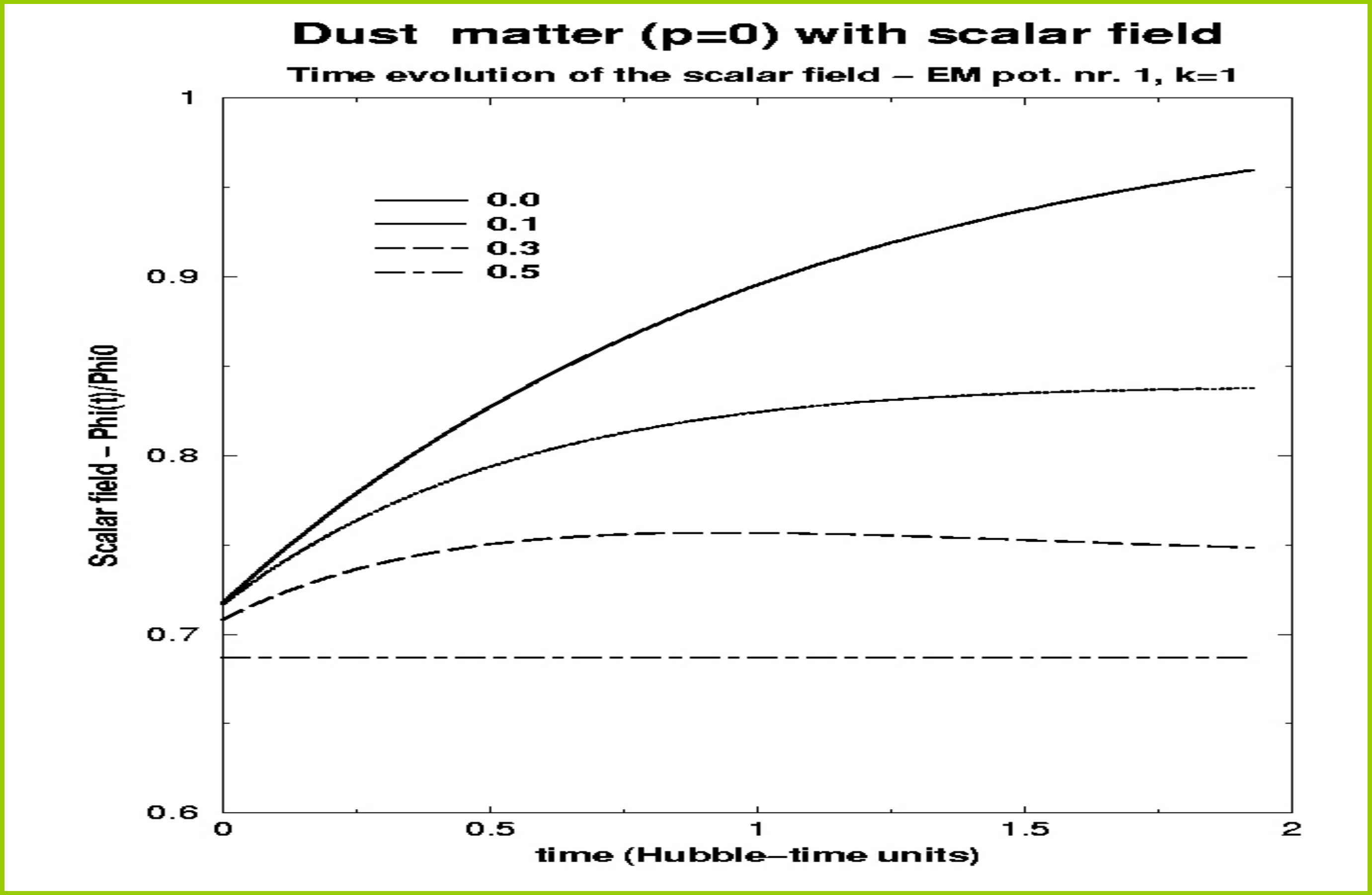
Convergence test for p=0 and Omega = 0.3

DeSItter exponential expansion



Hubble function time evolution

DeSItter exponential expansion



Scalar field time evolution

Tachionic potentials

Recently it has been suggested that the evolution of a tachyonic condensate in a class of string theories can have a cosmological significance (T. Padmanabham, Phys.rev. D, 66, 021301(R) 2002).
This theory can be described by an effective scalar field with a lagrangian of of the form

$$\mathcal{L} = -V(\phi)\sqrt{1+\partial_i\phi\partial^i\phi}$$

where the tachyonic potential has a positive maximum at the origin → $V(\phi) = V_0$ at $\phi = 0$

and has a vanishing minimum where the potential vanishes → $V(\phi) = 0$ at $\phi \longrightarrow +\infty$

Since the lagrangian has a potential, it seems to be reasonable to expect to apply successfully the method of ``reverse engineering'' for this type of potentials. As it was shown when we deal with spatially homogeneous geometry cosmology described with the FRW metric above we can use again a density and a negative pressure for the scalar field as

Tachyonic potentials

$$\rho_\phi = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}$$

and

$$p_\phi = -V(\phi)\sqrt{1-\dot{\phi}^2}$$

Now following the same steps as explained before we have the new Friedmann equations as :

$$\begin{aligned} 3H(t)^2 + 3K(t) &= \frac{4\pi V(\Omega+1)}{\sqrt{1-\dot{\phi}(t)^2}} \\ 3H(t)^2 + 3\dot{H}(t) &= \frac{4\pi V(\Omega+1-\frac{3}{2}(\Omega\gamma+\dot{\phi}(t)^2))}{\sqrt{1-\dot{\phi}(t)^2}} \end{aligned}$$

With matter included also. Here as usual we have $K(t) = k/R(t)^2$

Tachionic potentials

We also have a new Klein-Gordon equation, namely :

$$\frac{V\dot{\phi}(t)^2}{1-\dot{\phi}(t)^2} + 3VH(t)\dot{\phi}(t) + \frac{\partial V}{\partial \phi} = 0$$

All these results are then saved in a new library, cosmotachi.m which will replace the cosmo.m library we described in the previous lecture.

Now following the REM method we have finally :

$$\dot{\phi}^2 = \frac{2}{3}\frac{\bar{K}(t) - \dot{H}(t)}{\bar{K}(t) + H(t)^2}$$

$$V(t) = \frac{3}{8\pi}\sqrt{H(t)^2 + \frac{2}{3}\dot{H}(t) + \bar{K}(t)\sqrt{H(t)^2 + \bar{K}(t)}}$$

which we used to process different types of scale factor, same as in The Ellis-Madsen potentials above

Tachionic potentials

	$R(t)/R_0$	$\alpha = \phi(t) - \phi_0$	$V(\phi)$	
1.	$e^{\omega t}$	$-\sqrt{\frac{2}{3}}\frac{1}{\omega}tanh^{-1}\left(\sqrt{1+\frac{\omega^2}{k}e^{2\omega t}}\right)$	$\frac{3\omega^2}{8\pi}sinh(\sqrt{6}\omega\alpha)\times \times\sqrt{tanh(\frac{\sqrt{6}}{2}\omega\alpha)-\frac{2}{3}}$	
2.	$sinh(\omega t)$	$\frac{2\sqrt{6}}{3\omega}tan^{-1}(e^{\omega t})$	$-\frac{\omega^2\sqrt{3}}{2\pi}\sqrt{\frac{2+cos(\sqrt{6}\alpha\omega)}{1+cos(\sqrt{6}\alpha\omega)}}$	$k=0$
3.	t^n	$\sqrt{\frac{2}{3n}}t$	$\frac{1}{2\pi}\sqrt{1-\frac{2}{3n}\frac{n}{\alpha^2}}$	$k=0$
4.	t	$\sqrt{\frac{2}{3}}t$	$\frac{1}{2\sqrt{3}\pi}\frac{1+k}{\alpha^2}$	
5.	$sin(\omega t)$	$\pm\sqrt{\frac{3}{2}}ln\left(sec(\omega\alpha t)+tan(\omega\alpha t)\right)$	$\frac{\sqrt{6}\omega^2}{2\pi}e^{\sqrt{6}\omega\alpha t}\times\sqrt{\frac{1-e^{\sqrt{6}\omega\alpha t}-e^{-\sqrt{6}\omega\alpha t}}{(e^{\sqrt{6}\omega\alpha t}-1)^2}}$	$k=0$

Tachionic potentials. Here we denoted with R_0 the scale factor at the actual time t_0 and with α the quantity $\phi(t) - \phi_0$

1. J.P. Ostriker, P.J. Steinhardt, Nature, 377, 600 (1995)

2. S. Perlmutter et.al., Astrophys. J, 517, 565, (1999)

3. C.W. Misner, K.S. Thorne, J.A. Wheeler : Gravitation, Freeman, San Francisco, (1973)

4. J.N. Islam, An Introduction to Mathematical Cosmology, Cambridge University Press, Cambridge, (1992)

5. G.F.R. Ellis, M.S. Madsen, Class. Quantum Grav., 8,667 (1991)

6. G.F.R. Ellis, Murugan J., Tsagas C.G., Class. Quantum Grav. 21 233 (2004)

7. D.N. Vulcanov, V.D. Vulcanov, Analele Univ. de Vest din Timisoara, Seria Matematica-Informatica, vol. XLII, Fasc. special 2, p. 181 (2004), electronic version at ar.Xiv:cs.SC/0409006,

8. D.N. Vulcanov, IEEE Computer Society SYNASC06 Conference Post-proceedings, p. 47, (2006), electronic version at IEEE Digital Library and arXiv:gr-qc/0210006

9. D.N. Vulcanov, Central Europ. J. of Physics, vol. 6 (1) 2008, p. 84

End of part III and
The End

Thank you !